Control – S-domain

Kelvin Leung, Ph.D.

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http://www.kskelvin.net

Laplace Transform

- Laplace transform
  - Laplace transform is named after Pierre-Simon Laplace, who introduced the transform in his work on probability therory.
  - Laplace transform simplifies the process of analyzing the behavior of the system. In engineering applications, normally refer to s-domain, which corresponding to a linear time-invariant (LTI) system for system stability and dynamic analysis.
  - Laplace transform definition

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\text{where } s = \sigma + j\omega$$

- Reference: http://en.wikipedia.org/wiki/Laplace_transform

Mathematical Relationship

Commonly Used in Electronics

$$f(t) \leftrightarrow F(s)$$

$$\frac{df(t)}{dt} \leftrightarrow sF(s)$$

$$\int f(t)\,dt \leftrightarrow \frac{1}{s}F(s)$$

Initial value theorem :

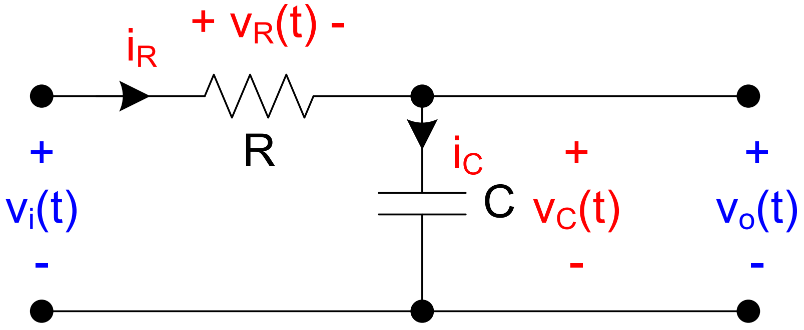
$$f(0^+) = \lim_{s\to\infty} sF(s)$$

Final value theorem :

$$f(\infty) = \lim_{s\to 0} sF(s)$$

Example of Laplace Transform in electronics circuit

Example of RC Filter
Question: Calculation Vo(s)/Vi(s)



$$(1): i_R = i_C$$

$$(2): i_C = C\frac{dv_c(t)}{dt} = C\frac{dv_o(t)}{dt}$$

For

$$v_i(t) = v_R(t) + v_C(t)$$

$$v_i(t) = i_R R + v_o(t)$$

$$v_i(t) = i_C R + v_o(t)$$

$$v_i(t) = CR\frac{dv_o(t)}{dt} + v_o(t)$$

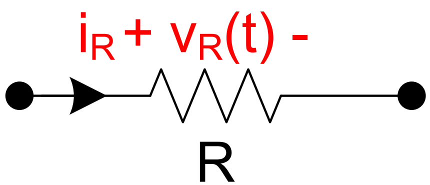
Laplace Transform

$$V_i(s) = sCRV_o(s) + V_o(s)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{sCR+1}$$

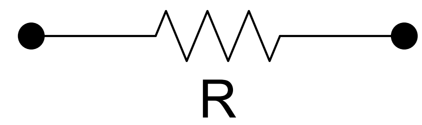
Question: Do we need to setup differential equation first???

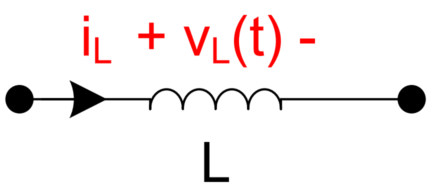
S-domain representation for circuit element



$$v_R(t) = Ri_R(t) \leftrightarrow V_R(s) = RI_R(s)$$

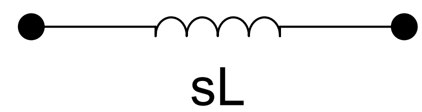
$$\therefore \frac{V_R(s)}{I_R(s)} = R$$

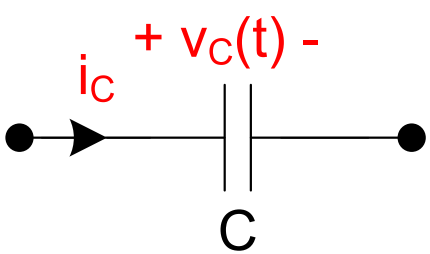




$$v_L(t) = L\frac{di_L(t)}{dt} \leftrightarrow V_L(s) = sLI_L(s)$$

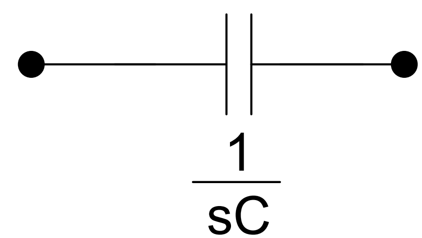
$$\therefore \frac{V_L(s)}{I_L(s)} = sL$$



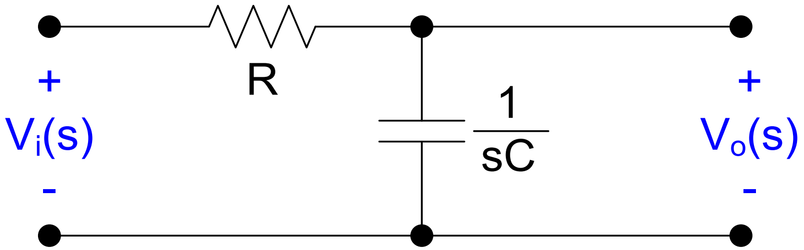


$$v_C(t) = \frac{1}{C}\int i_C(t)dt \leftrightarrow V_C(s) = \frac{1}{sC}I_C(s)$$

$$\therefore \frac{V_C(s)}{I_C(s)} = \frac{1}{sC}$$



Revisit RC example



$$V_o(s) = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} V_i(s)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sCR}$$

Benefit: Simplify circuit analysis without differential equation!

What can we do with the s-domain transfer function

Example: RC Filter

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{sCR+1} \rightarrow V_o(s) = \frac{1}{sCR+1} V_i(s)$$

If input is assumed to be unit step, i.e. $V_s(t) = \frac{1}{s}$

Apply Inital value theorem:

$$f(0^+) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s}{sCR+1} V_i(s)$$

$$f(0^+) = \lim_{s\to\infty} \frac{s}{sCR+1}\frac{1}{s} = \lim_{s\to\infty} \frac{1}{sCR+1} = 0$$

Apply Final value theorem

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{sCR+1} V_i(s)$$

$$f(\infty) = \lim_{s\to 0} \frac{s}{sCR+1}\frac{1}{s} = \lim_{s\to 0} \frac{1}{sCR+1} = 1$$

Steady state output independent of C and R

Laplace transform can help to calculate the steady state response without solving complicated differential equation.

Input Test Signal and Corresponding Laplace s-domain

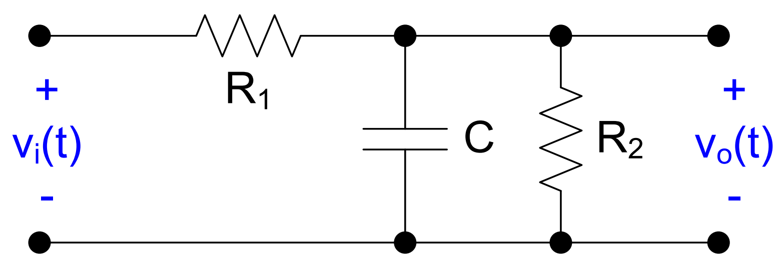
Function	Time domain $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace s-domain $F(s) = \mathcal{L}\{f(t)\}$
unit impulse	$\delta(t)$	1
delayed impulse	$\delta(t-\tau)$	$e^{-\tau s}$
unit step	$u(t)$	$\frac{1}{s}$
delayed unit step	$u(t-\tau)$	$\frac{e^{-\tau s}}{s}$
ramp	$t \cdot u(t)$	$\frac{1}{s^2}$

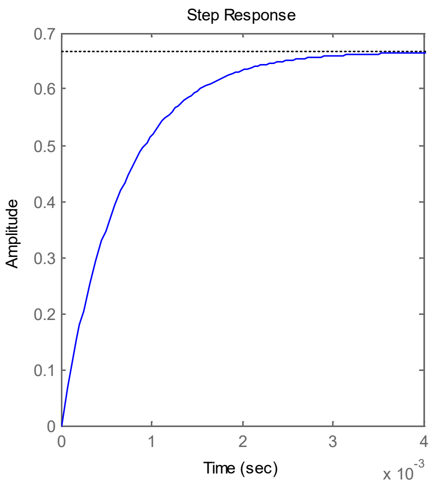
Benefit of s-domain

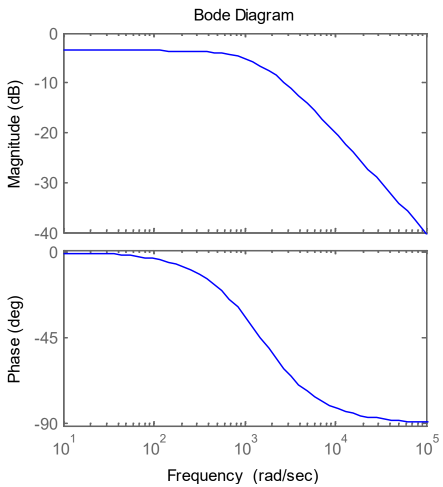
- In control theory, it well develop the understanding of 1st-order and 2st-order transfer function in s-domain. Therefore, without solving the equation, we can simply conclude the response of a system transfer function without solving the exact equation, and to design proper compensation network. This topic is discussed in another document
  - Control - System Response.doc
  - Control - Matlab and Control.doc

Homework – Question 1

- Question 1
  - Find system transfer function G(s)=Vo(s)/Vi(s)
  - Use final value theorem to determine steady state value if unit step response is used.
  - Assume R1=1k, R2=2k, C=1uF, use matlab to plot the step response and bode plot.
    - You will use matlab function
      - TF
      - STEP
      - BODE
    - Verify the step response with LTspice

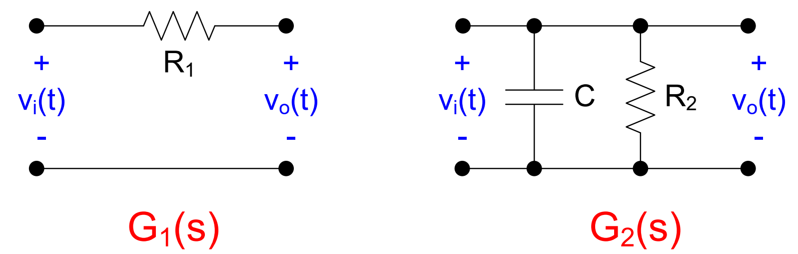


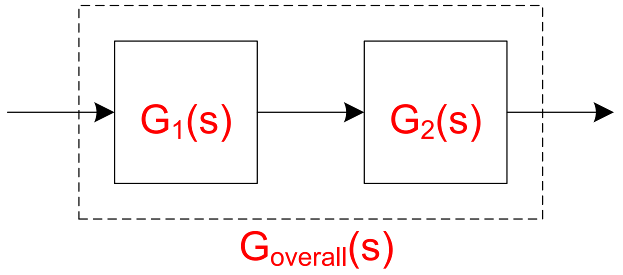




Homework – Question 2

- Question 2
  - Calculate system transfer function $G_1(s)$ and $G_2(s)$.
  - If $G_1(s)$ and $G_2(s)$ are connected in series (cascade), what is the new transfer function $G_{overall}(s)$?
  - Explain why $G_{overall}(s)$ is not same as answer of question 1.





Control – Block Diagram

Kelvin Leung, Ph.D.

09-20-2012

http://www.kskelvin.net

Block Diagram

Close Loop Transfer Function $\frac{V_{out}}{V_{in}}$

Eqn 1 : $V_{out} = G(s) \cdot V_{error}$

Eqn 2 : $V_f = H(s) \cdot V_{out}$

By $V_{error} = V_{in} - V_f$

$$\frac{V_{out}}{G(s)} = V_{in} - H(s) \cdot V_{out}$$

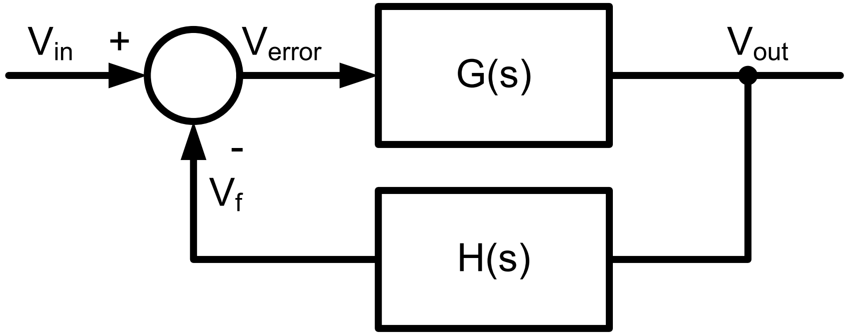
$$V_{in} = \left(\frac{1}{G(s)} + H(s) \right) V_{out} = \frac{1 + G(s)H(s)}{G(s)} \cdot V_{out}$$

$$\frac{V_{out}}{V_{in}} = \frac{G(s)}{1 + G(s)H(s)}$$

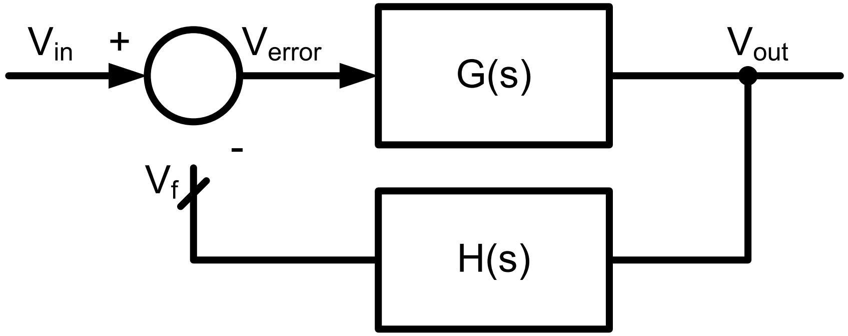
Open Loop Transfer Function $\frac{V_f}{V_{in}}$

$$\frac{V_f}{V_{in}} = G(s)H(s)$$

Close-Loop Transfer Function



Open-Loop Transfer Function



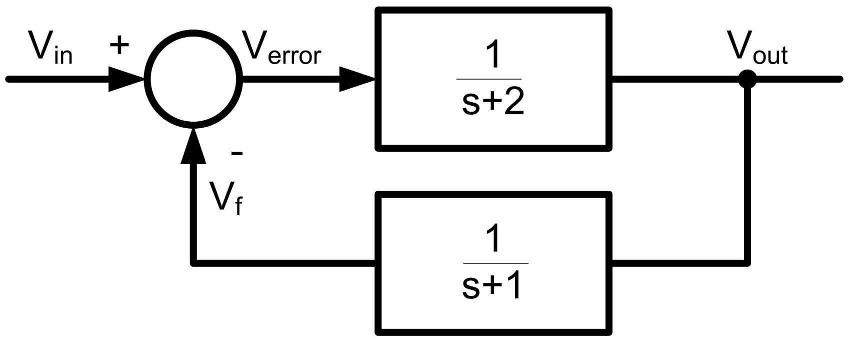
Numeral Example

Assume system transfer function called *T(s)*

$$\frac{V_{out}}{V_{in}} = T(s) = \frac{G(s)}{1+G(s)H(s)}$$

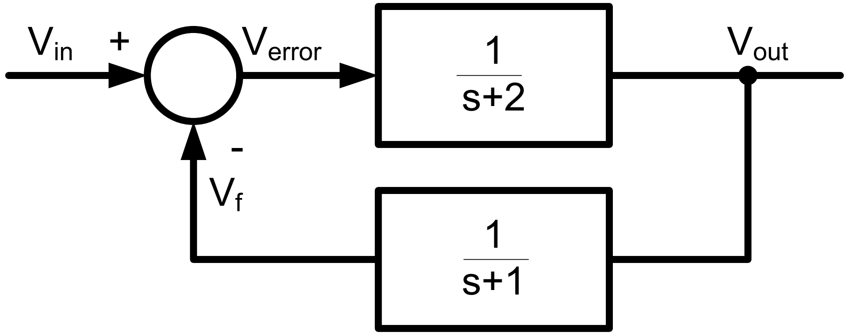
$$T(s) = \frac{\frac{1}{s+2}}{1+\frac{1}{s+2}\frac{1}{s+1}} = \frac{\frac{1}{s+2}}{\frac{(s+2)(s+1)+1}{(s+2)(s+1)}} = \frac{s+1}{1+(s+2)(s+1)}$$

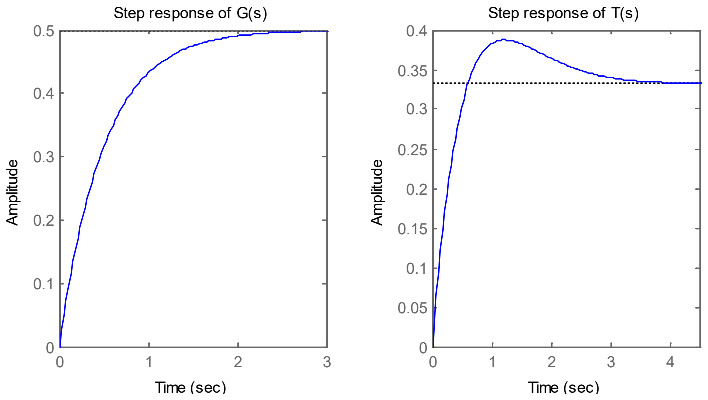
$$T(s) = \frac{s+1}{s^2+3s+3}$$



Howework #1

- Question
  - Use matlab to calculate the system transfer function T(s).
    - You will use matlab function
      - TF
      - FEEDBACK
  - Use matlab to plot the step response of G(s) and T(s).



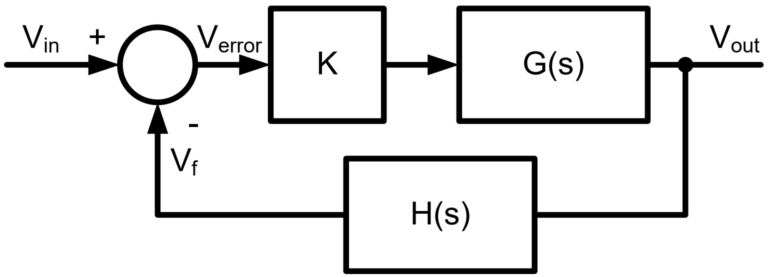


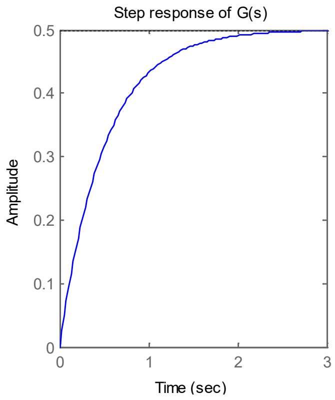
Homework #2

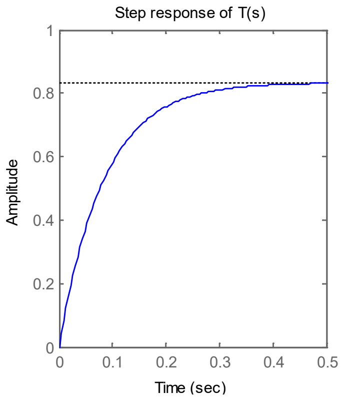
- Question
  - Assume

$$G(s) = \frac{1}{s+2},\ \ H(s) = 1, K = 10$$

    - Calculate T(s) with matlab.
    - Plot step response of G(s) and T(s).
      - What is the steady state value of T(s) in step response plot?
      - If K = 100, what is the new steady state value of T(s)?
        - What if the function of K?





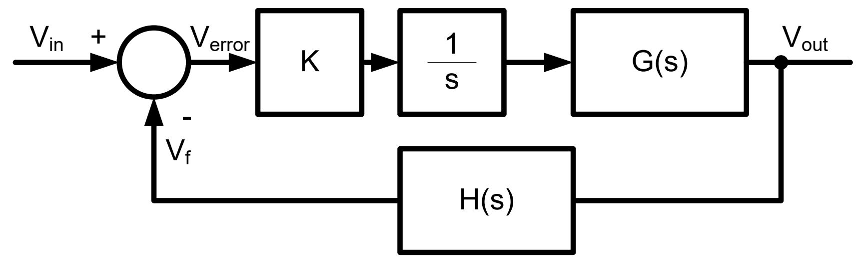


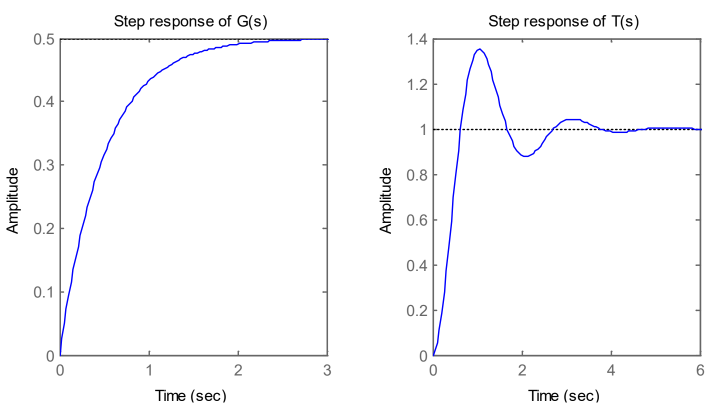
Howework #3

- Question
  - Assume

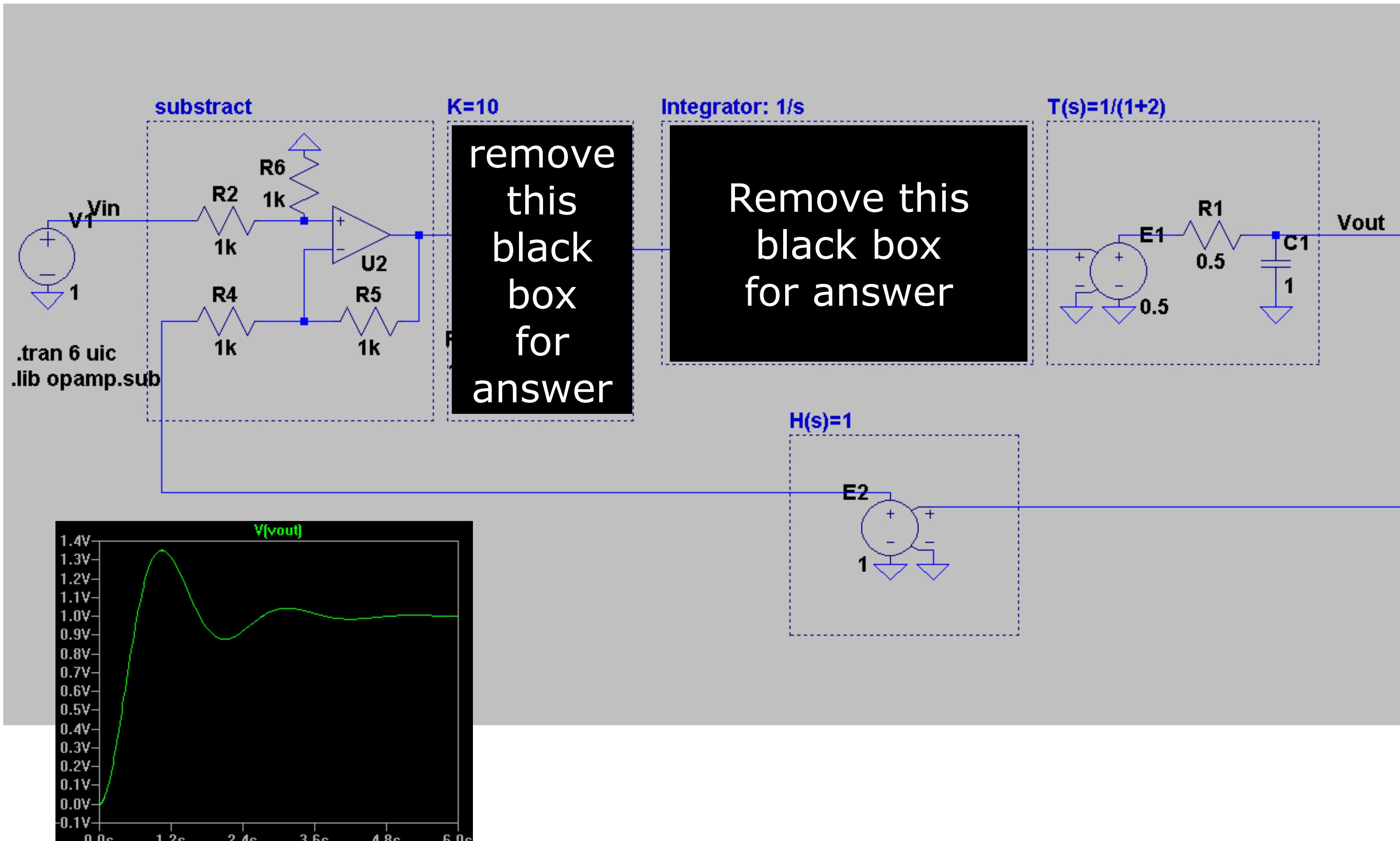
$$G(s)=\frac{1}{s+2},\ H(s)=1, K=10$$

- Calculate T(s) with matlab.
- Plot step response of G(s) and T(s).
  - What is the steady state value of T(s) in step response plot?
  - Change K to observe T(s) step response.
    - What if the function of s^{-1} in this system?
- In next slide, design op-amp compensation network in LTspice for this circuit. Use ans_blank.asc as template.





Homework #3



substract
K=10
remove this black box for answer
Integrator: 1/s
Remove this black box for answer
T(s)=1/(1+2)
V1
Vin
1
R2
1k
R6
1k
U2
R4
1k
R5
1k
.tran 6 uic
.lib opamp.sub
E1
0.5
R1
0.5
C1
1
Vout
H(s)=1
E2
1
V(vout)
1.4V
1.3V
1.2V
1.1V
1.0V
0.9V
0.8V
0.7V
0.6V
0.5V
0.4V
0.3V
0.2V
0.1V
0.0V
-0.1V
0.0s
1.2s
2.4s
3.6s
4.8s
6.0s

Control – 1^{st}-order and 2^{nd}-order system response

Kelvin Leung, Ph.D.

09-25-2012

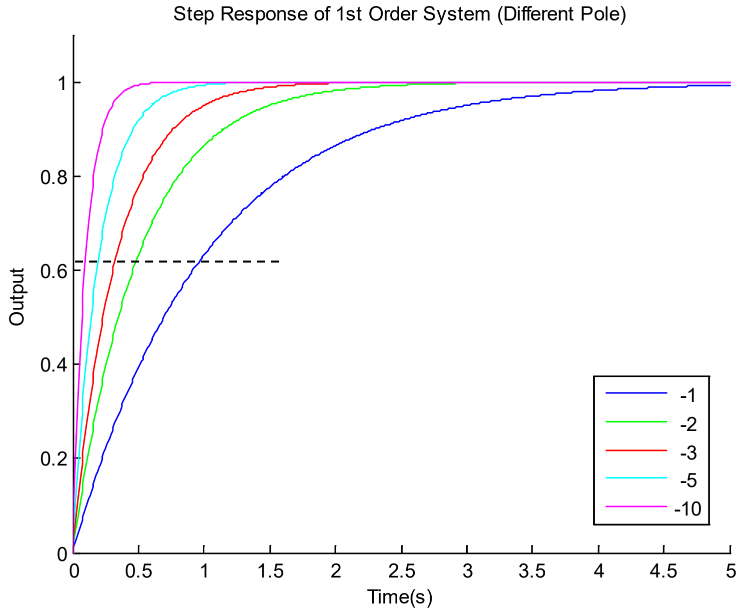
http://www.kskelvin.net

1^{st}-order System Step Response

- Assume 1^{st}-order system as

$$G(s) = \frac{p}{s+p}$$

- Pole is defined as the root of system denominator = 0
  - i.e. s + p = 0 → s = −p
- Observation
  - A less negative pole gives a slower system.
  - A positive pole gives an unstable system.
  - System Time-Constant = 1/p.
    - Time to achieve ~63% of output

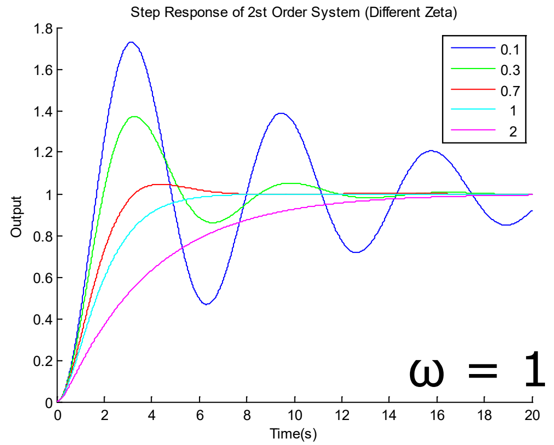


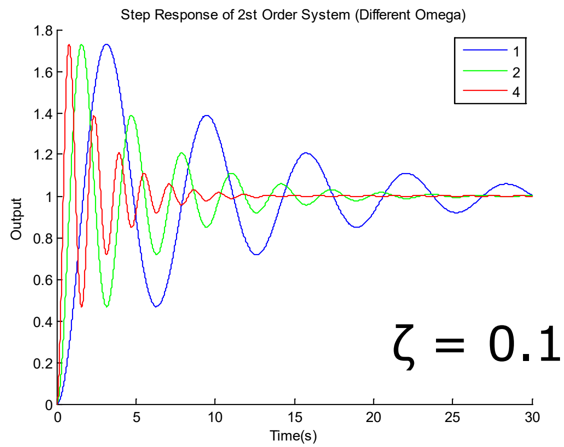
2^{nd}-order System Step Response

- Assume 2^{nd}-order system as

$$G(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

- Pole is defined as the root of system denominator = 0
  - i.e. $s^2 + 2\zeta\omega_n s + \omega_n^{\ 2} = 0$
- Zeta (ζ) Observation
  - ζ is damping ratio which affect overshoot and ringing.
  - ζ ~ 0.7 can minimize overshoot but maintain system speed
- Omega (ω_n) Observation
  - ω_n is natural frequency which affect system speed.
  - Changing ω_n doesn't affect overshoot or undershoot magnitude.





System Simplification

Assume 4th - order system

$$T(s)=\frac{1}{s^4+42s^3+483s^2+882s+802}$$

With the help of matlab "zpk"

$$T(s)=\frac{1}{802}\cdot\frac{401}{s^2+40s+401}\cdot\frac{2}{s^2+2s+2}$$

Approximation 1

$$T_1(s)=\frac{1}{802}\frac{401}{s^2+40s+401}$$

$$\omega_{n1}=\sqrt{401}\approx 20, \varsigma_1=0.999$$

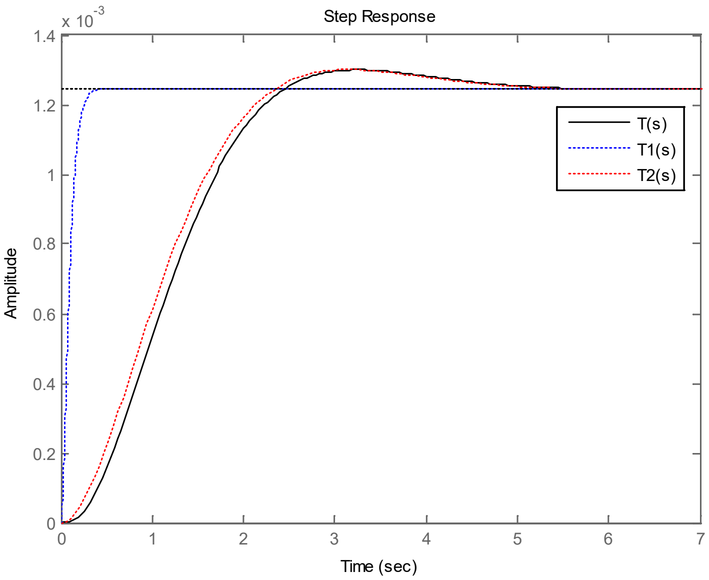
Approximation 2

$$T_2(s)=\frac{1}{802}\frac{2}{s^2+2s+2}$$

$$\omega_{n2}=\sqrt{2}\approx 1.414, \varsigma_2=0.707$$

As $\omega_{n2} << \omega_{n1}, T_2(s)$ can be used as approximation.

Smaller value represents slower system

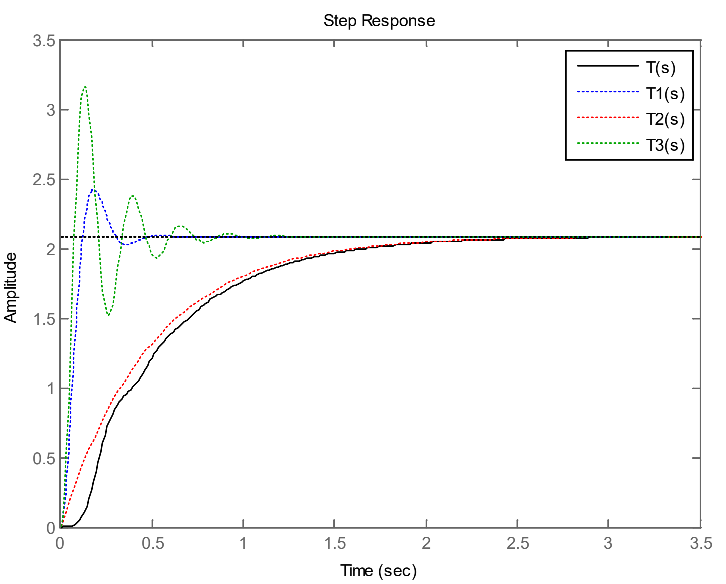


Homework 1

- Simplify the 5-th order system T(s)

$$T(s)=\frac{1000000}{s^5+32s^4+1260s^3+18400s^2+272000s+480000}$$

  - Help
    - Use matlab “tf” to create the transfer functions, then use “zpk” for factorization.
  - What is the approximate time-constant of the 5-th order system T(s)?
  - What is the purpose of System Simplification and the understanding of 1st and 2nd order system response?



Root-Locus Method

Kelvin Leung, Ph.D.

10-18-2012

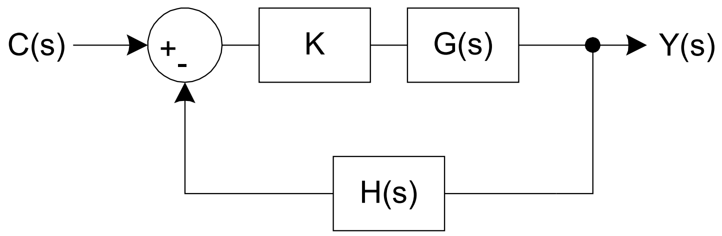
http://www.kskelvin.net

Fundamental of Root Locus

Close-loop system is defined as,

$$T(s)=\frac{KG(s)}{1+KG(s)H(s)}$$

where open-loop system is $G(s)H(s)$



Assume $G(s)=\dfrac{num_G}{den_G}$ and $H(s)=\dfrac{num_H}{den_H}$

Poles of $T(s)$ is the root of eqn $1+KG(s)H(s)=0$

$$\because 1+K\frac{num_G}{den_G}\frac{num_H}{den_H}=0$$

$$\therefore den_G den_H + K\cdot num_G num_H = 0$$

If $K=0 \Rightarrow den_G den_H = 0$

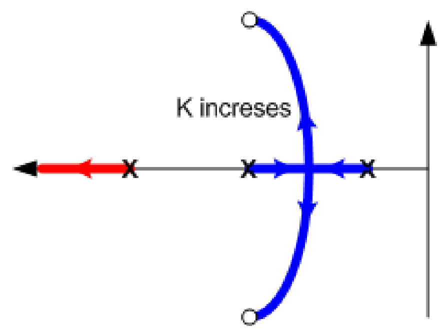
therefore, open-loop system poles are close-loop system poles at K = 0

If $K\rightarrow\infty \Rightarrow num_G num_H = \lim_{K\rightarrow\infty}\dfrac{-den_G den_H}{K}=0$

therefore, open-loop system zeros are close-loop system poles at $K\rightarrow\infty$

Therefore, determine the trajectories of $1+KG(s)H(s)=0$ for $K=0$ to ∞ is the locus of close-loop system poles.

Root Locus: Trajectories of Close-Loop System Poles



X Poles of Open-Loop System, Poles of Close-Loop System @ K = 0
O Zeros of Open-Loop System, Poles of Close-Loop System @ K = Inf

Remark:
Poles of G(s)H(s) can force $den_G=0$ or $den_H=0$
Zeros of G(s)H(s) can force $num_G=0$ or $num_H=0$

Matlab of Root Locus

Matlab Code

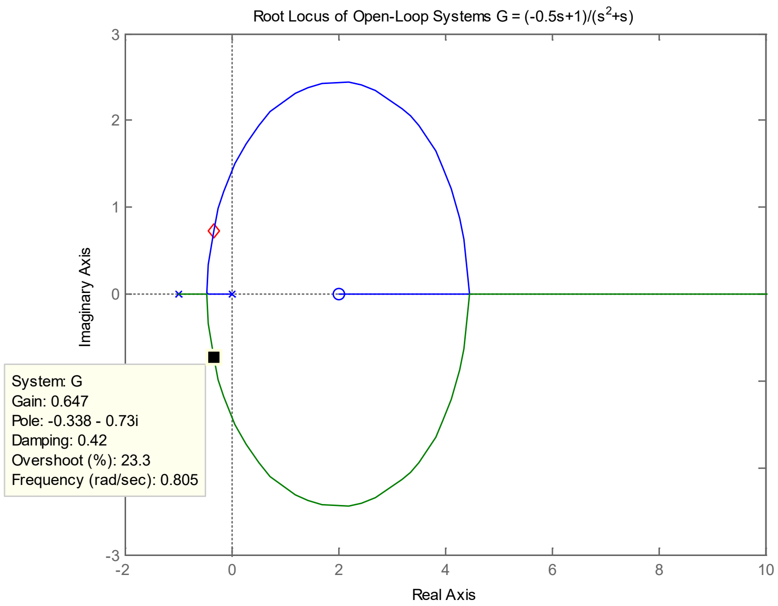
```
% define the open-loop system as G = (-0.5s+1)/(s^2+s)
num=[-0.5 1];
den=[1 1 0];
G = tf(num,den);
```

$$\boxed{\text{Open - Loop Transfer Function as} \quad G(s)=\frac{-0.5s+1}{s^2+s}}$$

```
% calculate pole and zero of close-loop systems
K = 0.647;
T = feedback(K*G,1);
[p,z]=pzmap(T);
plot(real(p),imag(p),'rd'); hold on;
```

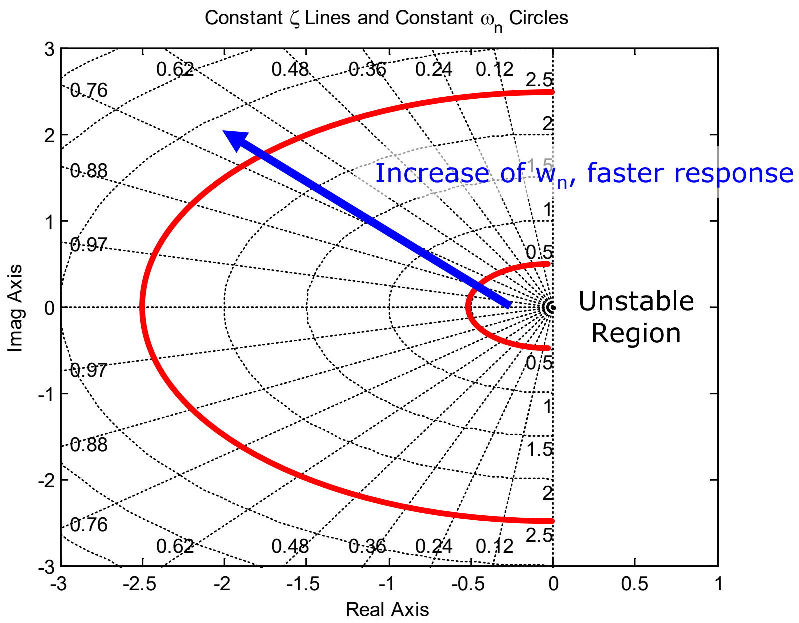
$$\boxed{\text{Calculate Close - Loop Transfer Function as} \quad T=\frac{KG(s)}{1+KG(s)}}$$

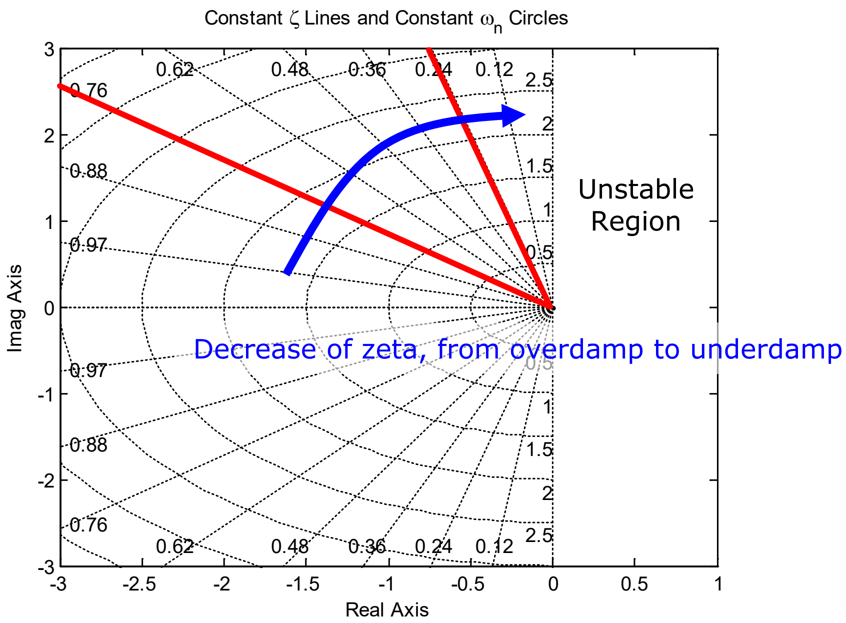
```
% plot root-locus of open-loop systems
rlocus(G)
title('Root Locus of Open-Loop Systems G = (-
      0.5s+1)/(s^2+s)');
```



Prove from this matlab routine:
rlocus plots the root locus of open-loop system,
which represents the locus of poles of close-loop system with K from 0 to INF.

Root Locus Plot (sgrid)



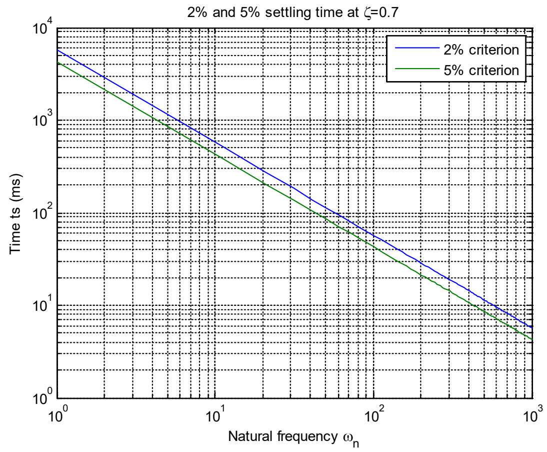


System dynamic design criteria

ω_n is natural frequency

ζ is damping ratio

where $\sigma = \omega_n \zeta$, $\omega_d = \omega_n \sqrt{1-\zeta^2}$



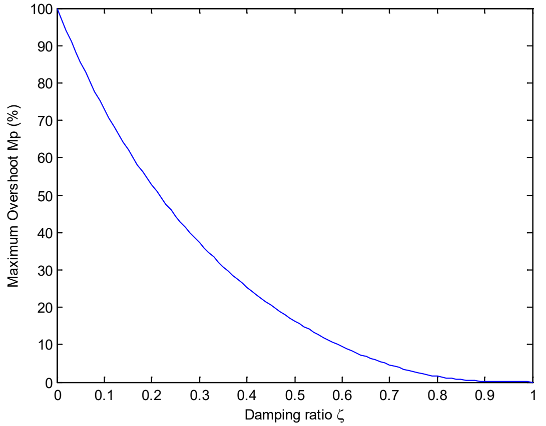
Settling time (t_S)

$$2\% \text{ criterion}: \; t_S = 4T = \frac{4}{\sigma} = \frac{4}{\omega_n \zeta}$$

$$5\% \text{ criterion}: \; t_S = 3T = \frac{3}{\sigma} = \frac{3}{\omega_n \zeta}$$

Maximum overshoot (M_p)

$$M_p = e^{-\frac{\sigma}{\omega_d}\pi} = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}$$



Design with Root-Locus Method

- Matlab Code

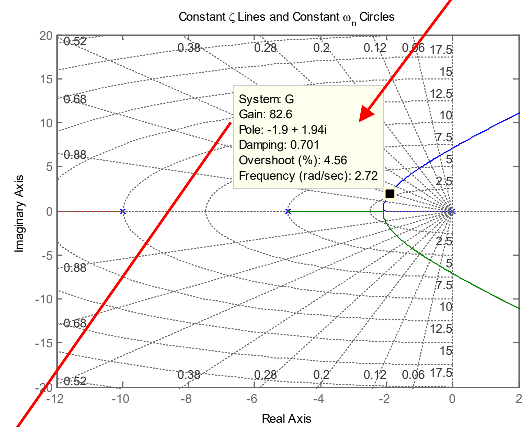
```
% define the open-loop system as G(s)
z=[];
p=[0 -5 -10];
k=1;
G = zpk(z,p,k);
```

Open - Loop Transfer Function as

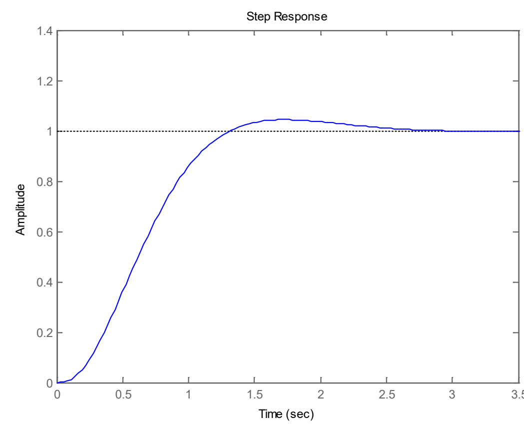
$$G(s) = \frac{1}{s(s+5)(s+10)}$$

```
% plot root-locus of open-loop systems
figure;
K=[0:0.2:4e3];
rlocus(G,K); sgrid;
axis([-12 2 -20 20]);
title('Constant \zeta Lines and Constant
     \omega_n Circles');

% form close-loop systems T(s)
K = 82.6;
T = feedback(K*G,1);

% plot step response of close-loop system
figure;
step(T);
```

(1) In root locus plot, put a marker and search for zeta = 0.7



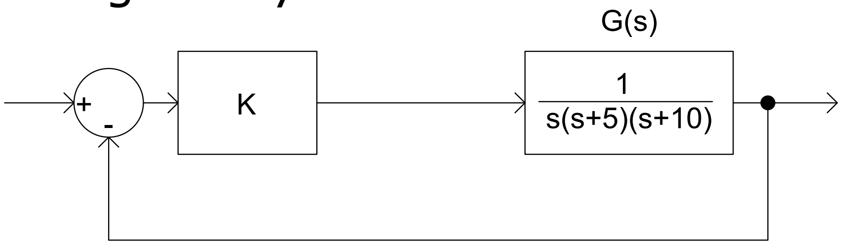
(2) Gain=82.6 for zeta = 0.7



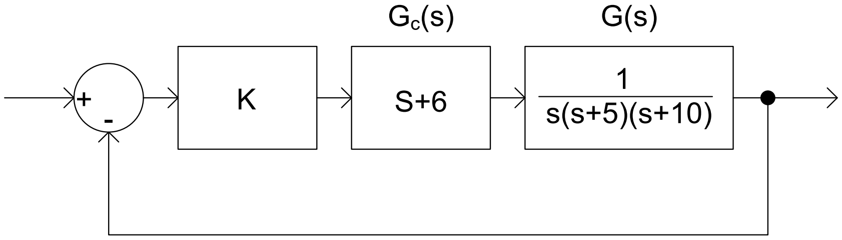
Design with Root-Locus Method (Improve response with addition zeros)

- Design of compensator
  - Reference
    - P.310 of Modern Control Engineering (5th Edition), Katsuhiko Ogata.
  - Effects of the addition of poles
    - Pulling the root locus to the right, tending to lower the system's relative stability and to slow down the settling of the response.
  - Effects of the addition of zeros
    - Pulling the root locus to the left, tending to make the system more stable and to speed up the settling of the response.
- Matlab example
  - Based on previous design, we add a compensator with zero = -6.

Original System

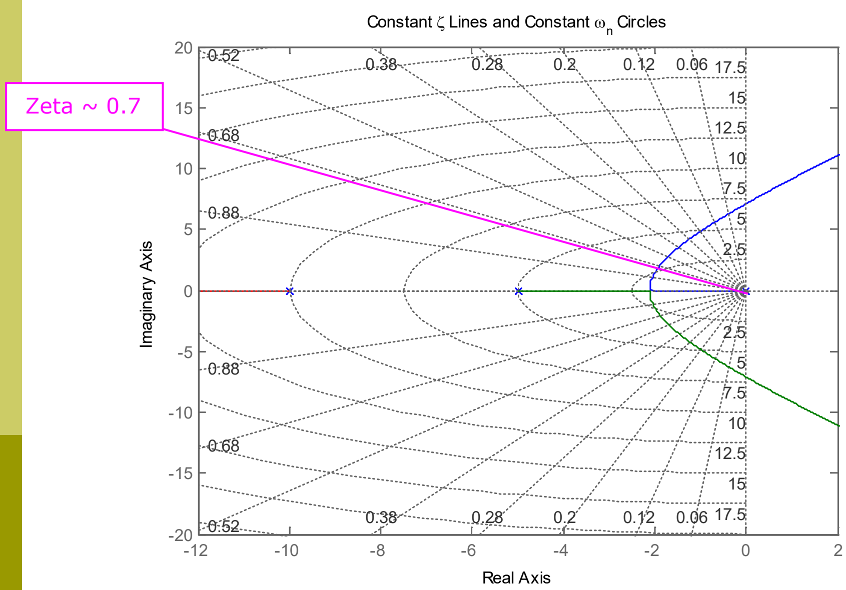


System with addition zero

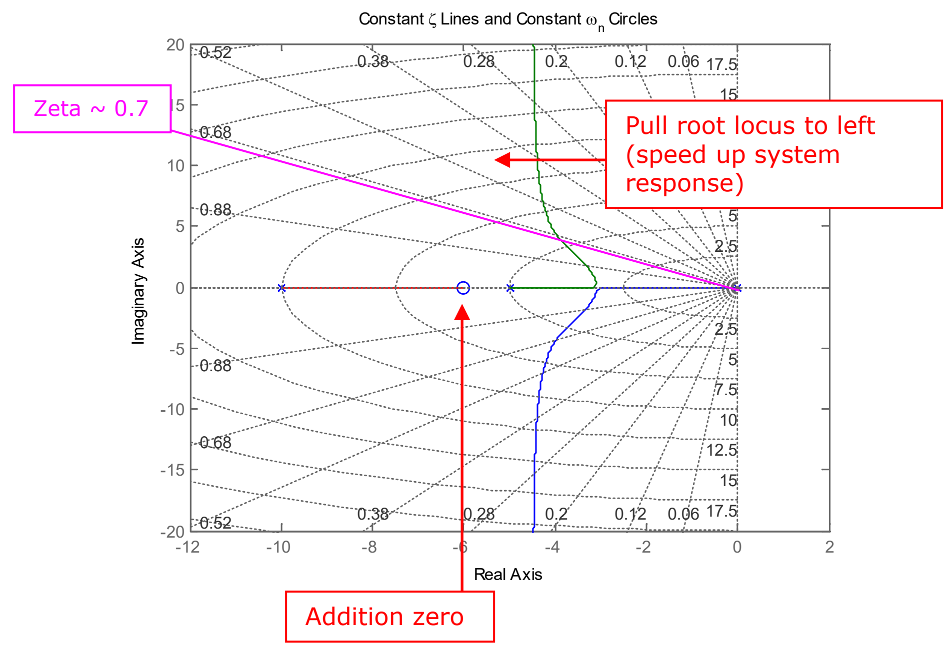


Design with Root-Locus Method (Improve response with addition zeros)

Original System



System with addition zero

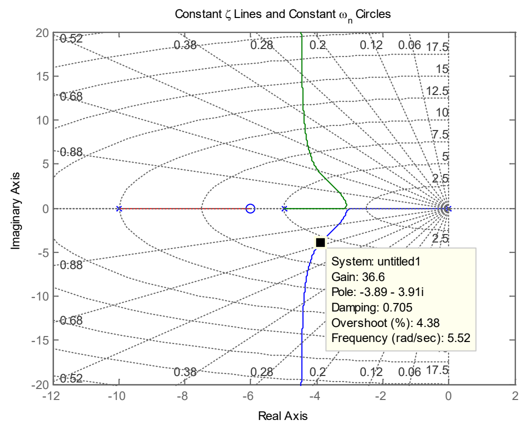


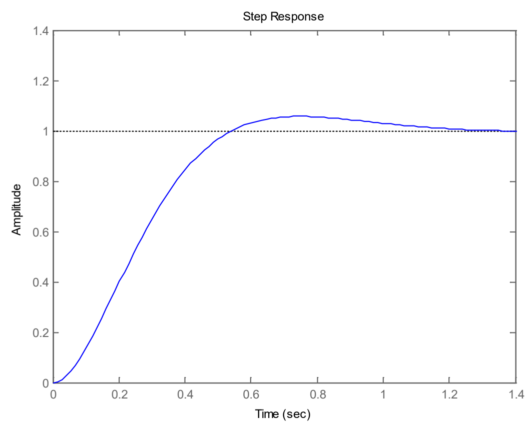
Observation
- Almost double w_n at zeta = 0.7
- Close-loop stable for all K.

Design with Root-Locus Method (Improve response with addition zeros)

- Matlab Code

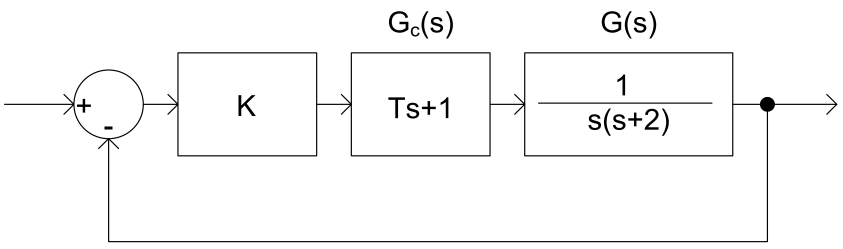
```
% define the open-loop system as G(s)
z=[];
p=[0 -5 -10];
k=1;
G = zpk(z,p,k);

% compensator transfer function Gc(s)
num=[1 6];
den=[1];
Gc = tf(num,den);

% plot root-locus of open-loop systems Gc(s)G(s)
figure;
K=[0:0.2:1e3];
rlocus(Gc*G,K); sgrid;
axis([-12 2 -20 20]);
title('Constant \zeta Lines and Constant \omega_n
      Circles');

% form close-loop systems T(s)
K = 36.6;
T = feedback(K*Gc*G,1);

% plot step response of close-loop system
figure;
step(T);
```

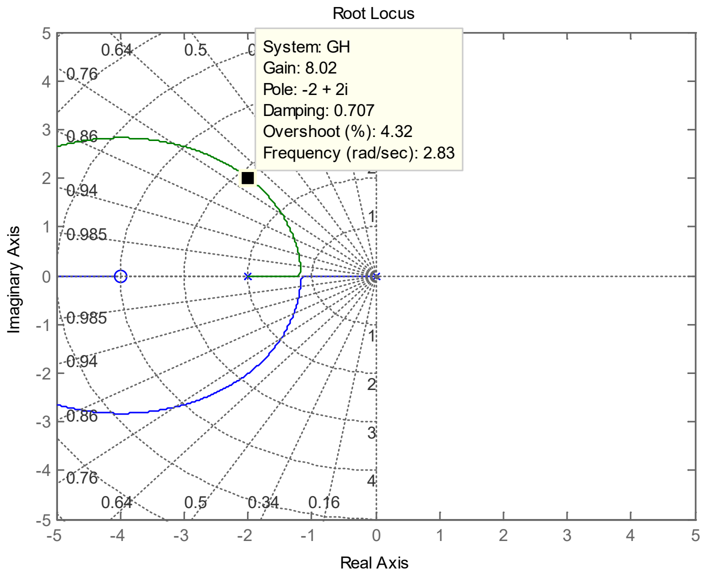




Exercise #1

- Exercise #1
  - Use matlab to determine the gain K and time constant T of the controller Gc(s) such that the closed-loop poles are located at s=-2+/-j2.
    - Hint: T can be determined by trial and error method in matlab.





Rules for Constructing Root Locus

Kelvin Leung, Ph.D.

10-19-2012

http://www.kskelvin.net

Rules for Constructing Root Loci

- Reference
  - P.283-287, "Modern Control Engineering", Fifth Edition, Katsuhiko Ogata
  - The construction rules in this ppt follows the Ogata textbook.

Root Locus is the root trajectory of Characteristic Equation

$$den_G den_H + K \cdot num_G num_H = 0$$

or

$$1 + K \cdot G(s)H(s) = 0$$

or represented in general form

$$B(s) + K \cdot A(s) = 0$$

where

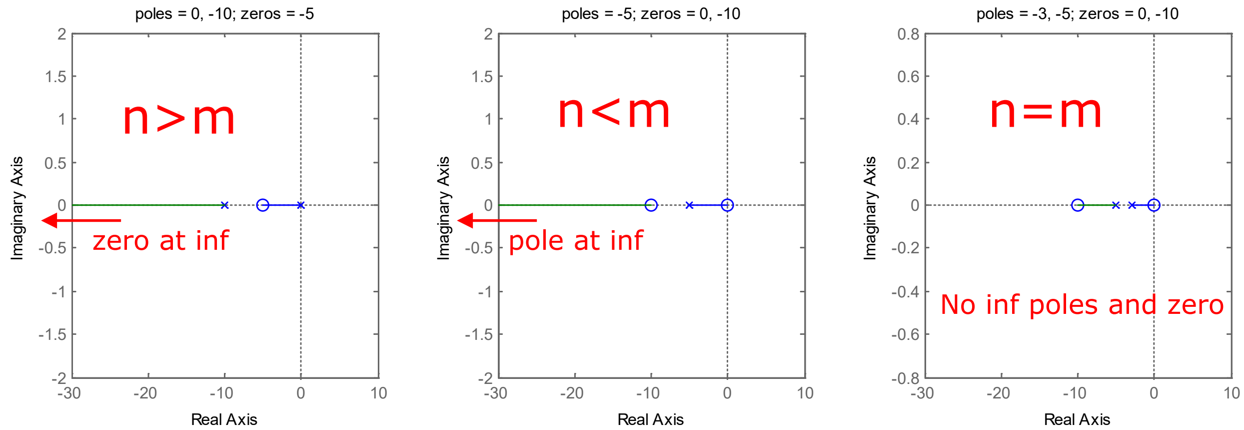
roots of $A(s) = 0$ are open-loop system zeros

roots of $B(s) = 0$ are open-loop system poles

Therefore, root locus can apply for any system which re-write to this general form.

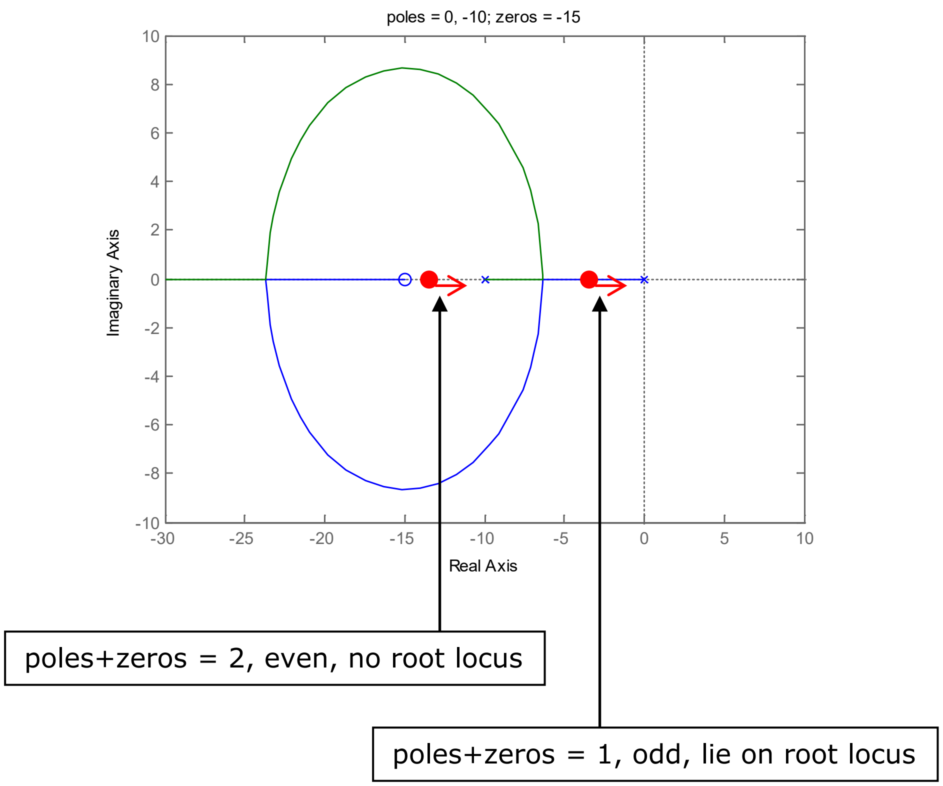
Rule #1

- Locate the poles and zeros of G(s)H(s) on the s plane.
  - Root-locus branches start from open-loop poles and terminate at zeros (finite zeros or zeros at infinity)
  - Assume
    - Number of poles = n
    - Number of zeros = m
      - If n>m, then system has n-m infinity zeros
      - If n<m, then system has m-n infinity poles
      - If n=m, then system has no infinity poles or zeros.

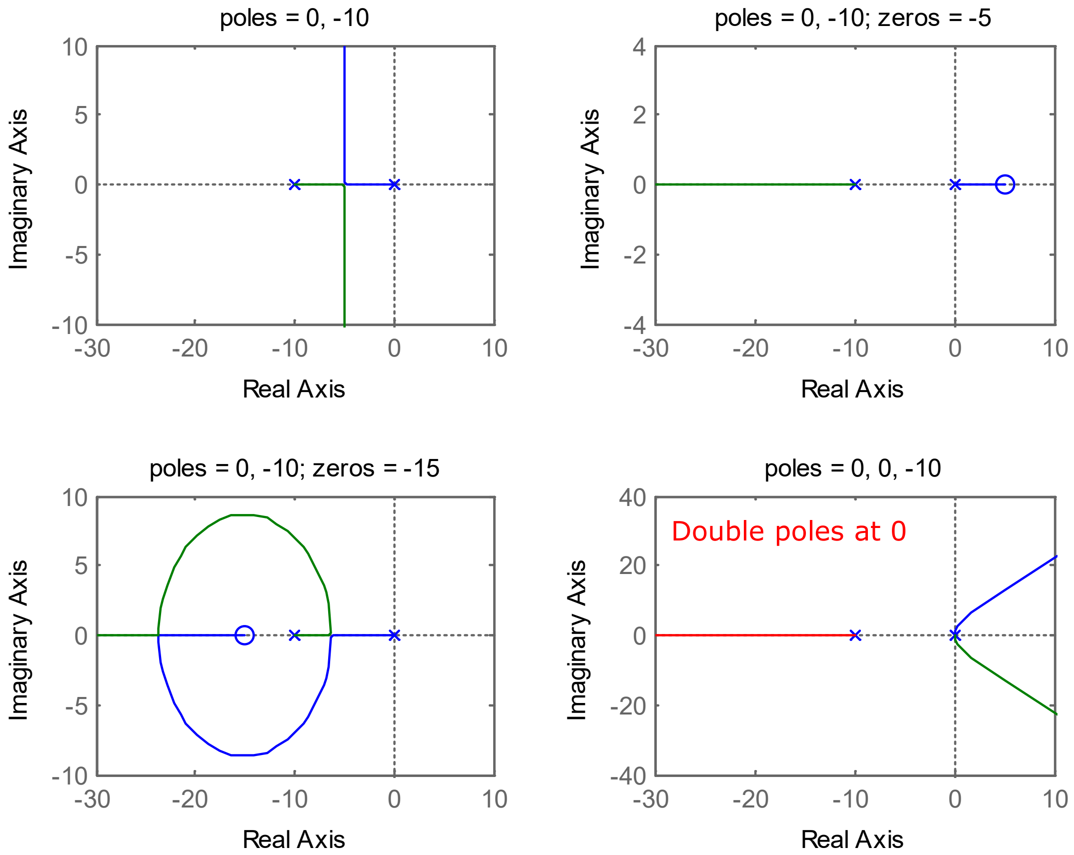


Rule #2

- Determine the root loci on the real axis
  - Put a test point on the real axis,
    - If the total poles and zeros to the right of this test point is odd, then this point lies on the root locus, otherwise, point doesn’t lies on the root locus.



Rule #2 (examples)



Rule #3

- Determine the asymptotes of root loci

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ (2k+1)}{n-m} \qquad (k = 0,1,2,\ldots)$$

where

n = number of finite poles of $G(s)H(s)$

m = number of finite zeros of $G(s)H(s)$

$$\text{If open-loop system is } G(s)H(s) = \frac{\prod (s+z_n)}{\prod (s+p_n)}$$

Intersection of asymptotes

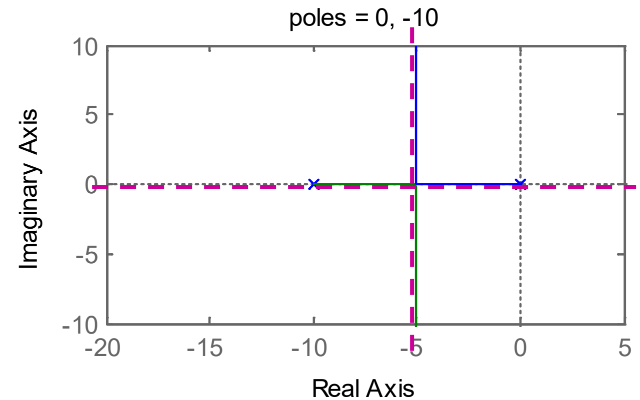
$$s_{\text{intersection}} = -\frac{\sum p_n - \sum z_n}{n-m}$$

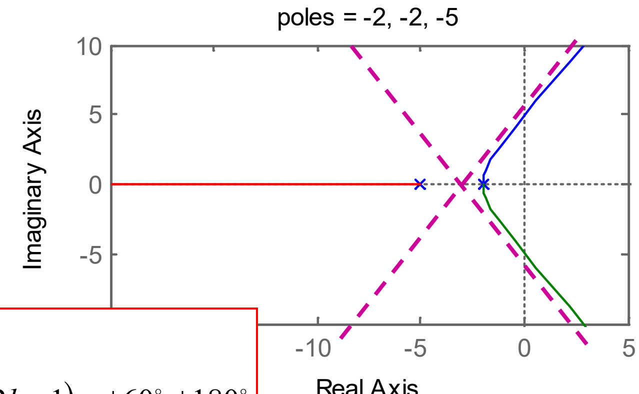
Caution p_n and z_n not poles and zeros
For example, if poles are -1, -2, p_1=1, p_2=2
$(s+p_1)(s+p_2)$

$n = 2, m = 0$

$$\text{angles of asymptotes} = \frac{\pm 180^\circ (2k+1)}{2} = \pm 90^\circ (2k+1) = \pm 90^\circ, \pm 180^\circ$$

$$s_{\text{intersection}} = -\frac{\sum p_n - \sum z_n}{n-m} = -\frac{(0+10)-0}{2} = -5$$





$n = 3, m = 0$

$$\text{angles of asymptotes} = \frac{\pm 180^\circ (2k+1)}{3} = \pm 60^\circ (2k+1) = \pm 60^\circ, \pm 180^\circ$$

$$s_{\text{intersection}} = -\frac{\sum p - \sum z}{n-m} = -\frac{(2+2+5)-0}{3} = -3$$

Rule #4

- Find the breakaway and break-in points

Identify the characteristic equation in this format

$$B(s) + K \cdot A(s) = 0$$

Breakaway or Break - in points are the roots of

$$\frac{dK}{ds} = -\frac{A(s)\frac{dB(s)}{ds} - B(s)\frac{dA(s)}{ds}}{\left(A(s)\right)^2} = 0$$

$$\therefore A(s)\frac{dB(s)}{ds} - B(s)\frac{dA(s)}{ds} = 0$$

A actual breakaway or break - in point can obtain K as a positive number

by substitute that s (root of $\frac{dK}{ds} = 0$) into characteristic equation.

Rule #4

Assume Open - Loop system

$$G(s)H(s)=\frac{s+15}{s(s+10)}=\frac{s+15}{s^2+10s}$$

$\therefore$ Characteristic equation is

$$1+K\cdot G(s)H(s)=0\Rightarrow 1+K\frac{s+15}{s^2+10s}=0$$

$$\therefore (s^2+10s)+K(s+15)=0$$

where $A(s)=s+15, B(s)=s^2+10s$

Breakaway or break - in points

$$\text{By } \frac{dK}{ds}=-\frac{A(s)\frac{dB(s)}{ds}-B(s)\frac{dA(s)}{ds}}{(A(s))^2}=0$$

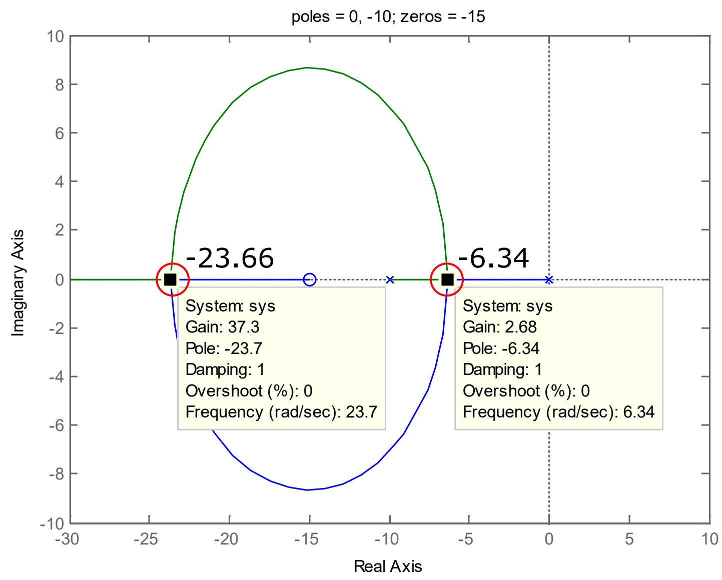
$$\Rightarrow A(s)\frac{dB(s)}{ds}-B(s)\frac{dA(s)}{ds}=0$$

$$\Rightarrow (s+15)\frac{d(s^2+10s)}{ds}-(s^2+10s)\frac{d(s+15)}{ds}=0$$

$$\Rightarrow (s+15)(2s+10)-(s^2+10s)(1)=0$$

$$\Rightarrow s^2+30s+150=0$$

$$\therefore s=-6.34 \text{ or } s=-23.66$$



Put $s=-6.34$ into $(s^2+10s)+K(s+15)=0\Rightarrow K=2.68$

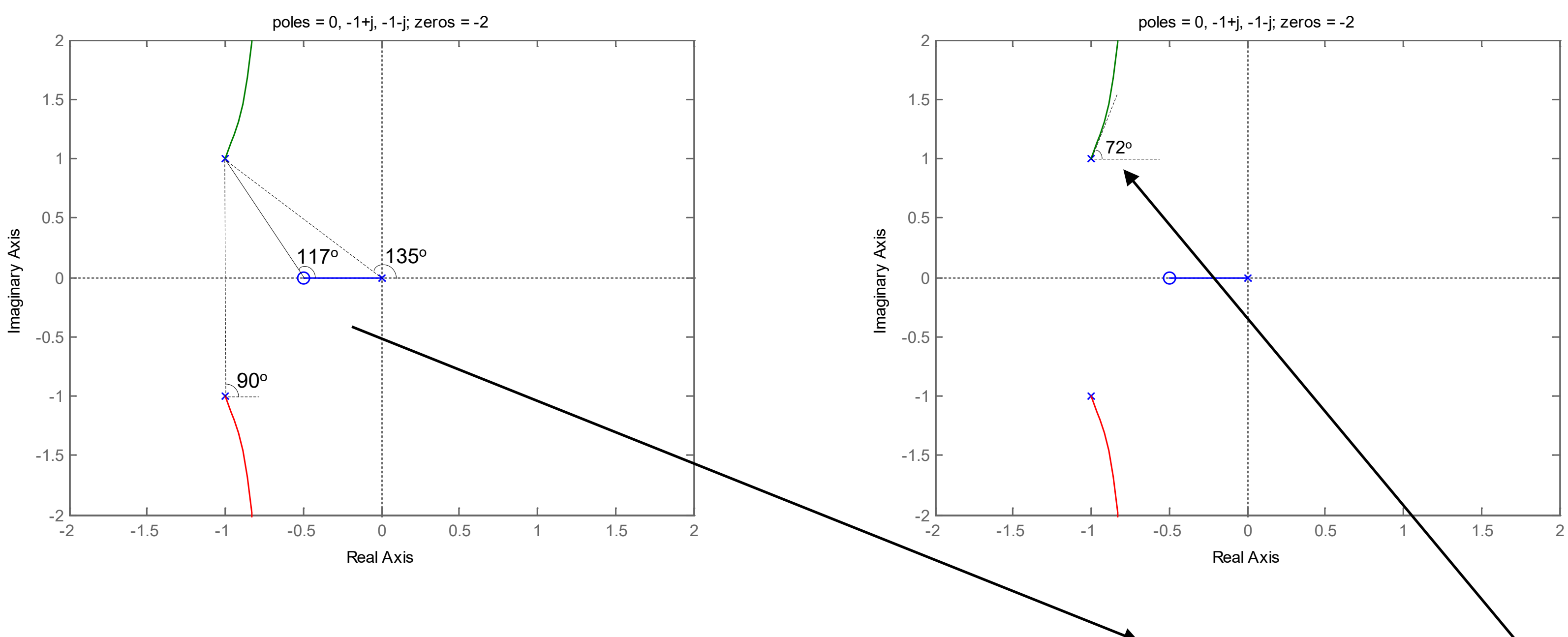
Put $s=-23.66$ into $(s^2+10s)+K(s+15)=0\Rightarrow K=37.32$

Rule #5

- Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero)

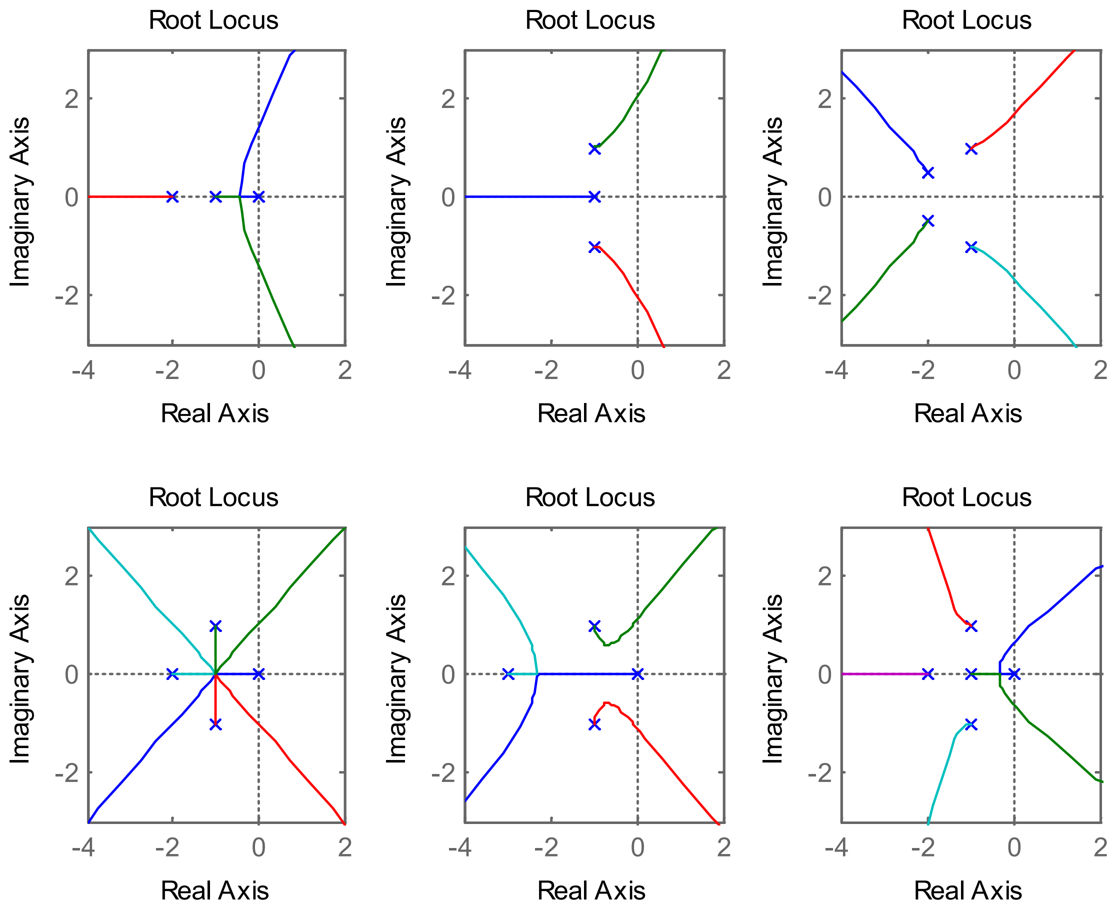
  - Angle of departure from a complex pole = 180^o
    – (sum of the angles of vectors to a complex pole in question from other poles)
    + (sum of the angles of vectors to a complex pole in question from zeros)
  - Angle of arrival at a complex zero = 180^o
    – (sum of the angles of vectors to a complex zero in question from other zeros)
    + (sum of the angles of vectors to a complex zero in question from poles)

Rule #5 (example)

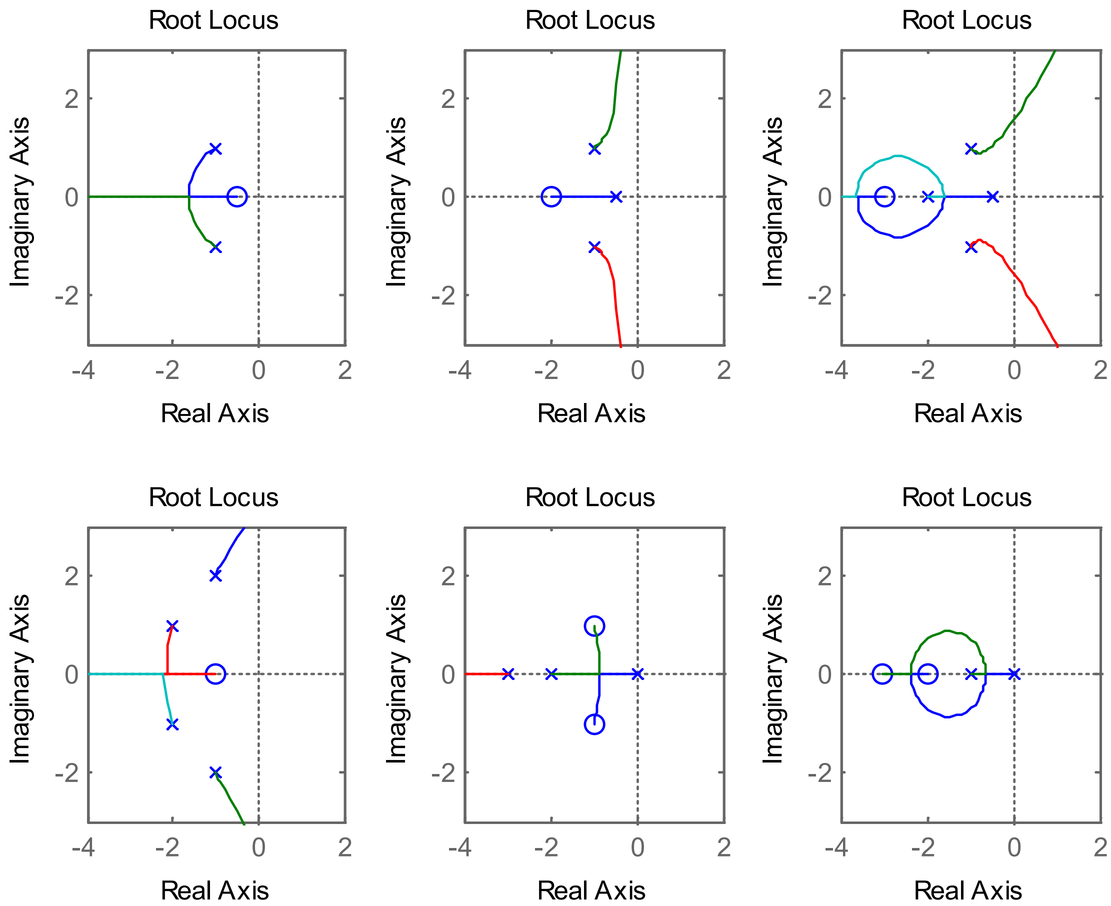


$$\text{Angle of departure from a complex pole} = 180^\circ - \left(135^\circ + 90^\circ\right) + \left(117^\circ\right) = 72^\circ$$

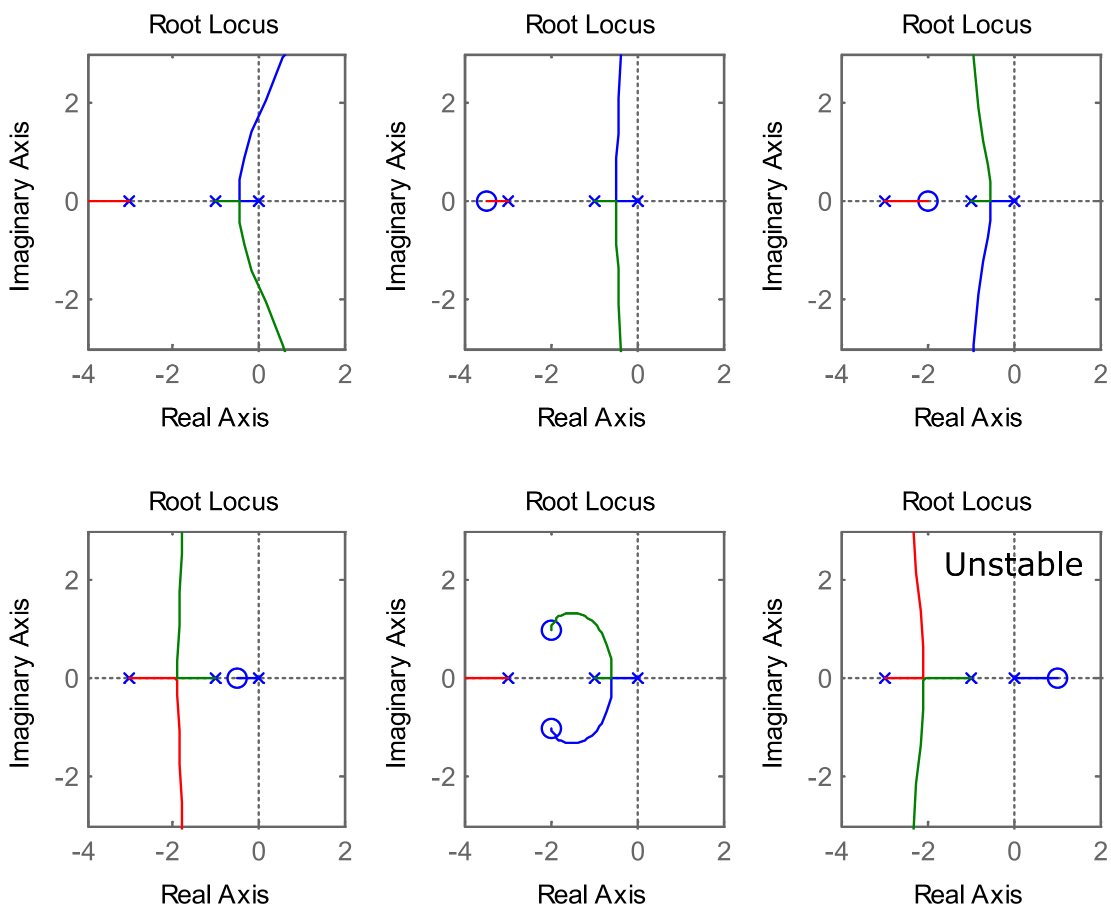
Open-Loop Pole-Zero Configurations and the Corresponding Root Loci



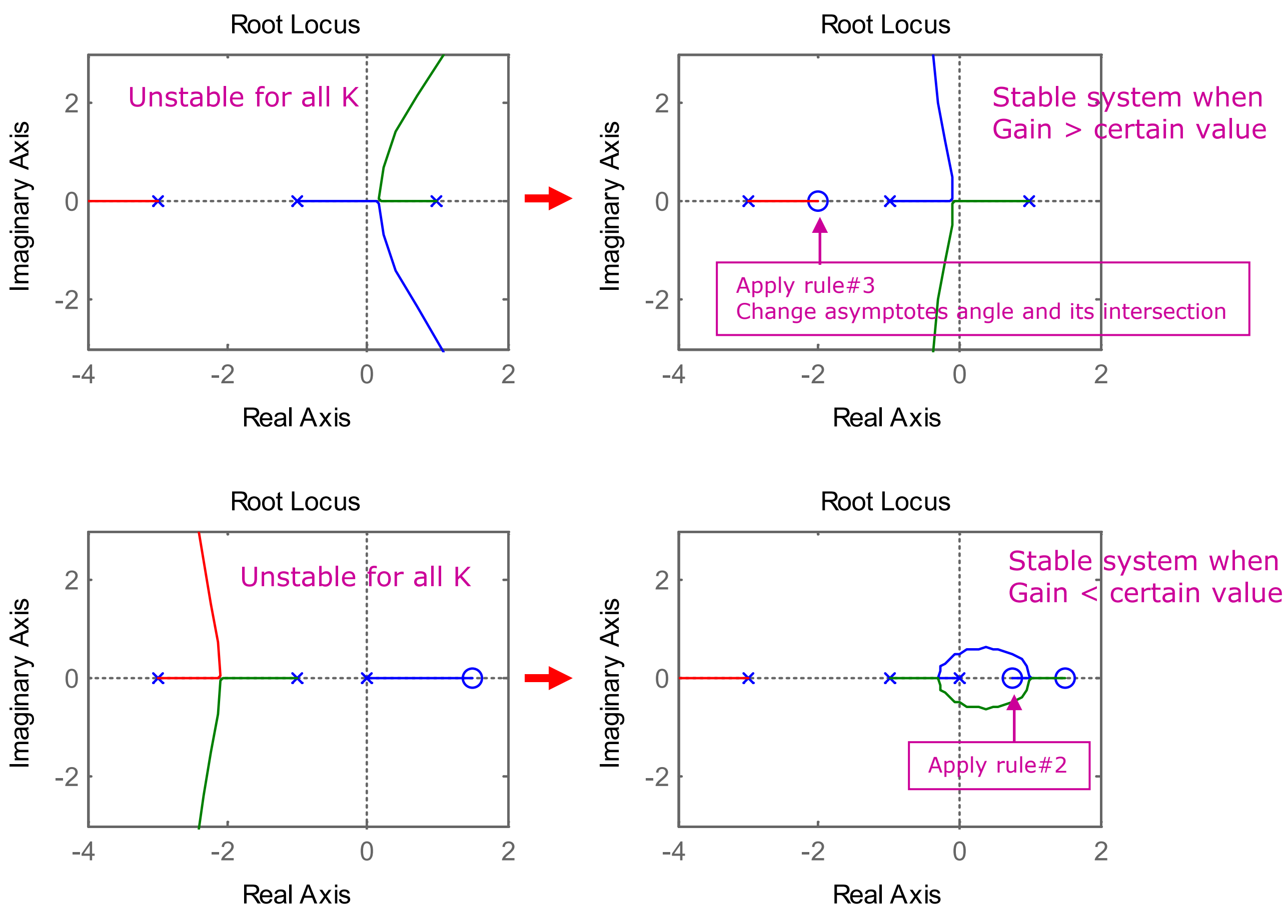
Open-Loop Pole-Zero Configurations and the Corresponding Root Loci



Example of compensating a system with addition zeros



Example of compensating an unstable system



Frequency Response Method - Bode Plot

Kelvin Leung, Ph.D.

10-22-2012

http://www.kskelvin.net

Bode Plot

- Bode Plot
  - Presenting frequency-response characteristics in graphical forms.
  - Plot of Logarithm of the magnitude of a sinusoidal transfer function
  - Plot of phase angle
  - Against the frequency on a logarithmic scale.

System transfer function $G(s)$

Substitute $s = j\omega = j2\pi f$

$G(s)$ can be expressed as

$$G(j\omega) = |G(j\omega)| \angle G(j\omega)$$

Logarithm Magnitude

$$|G(j\omega)|_{dB} = 20 \log(|G(j\omega)|)$$

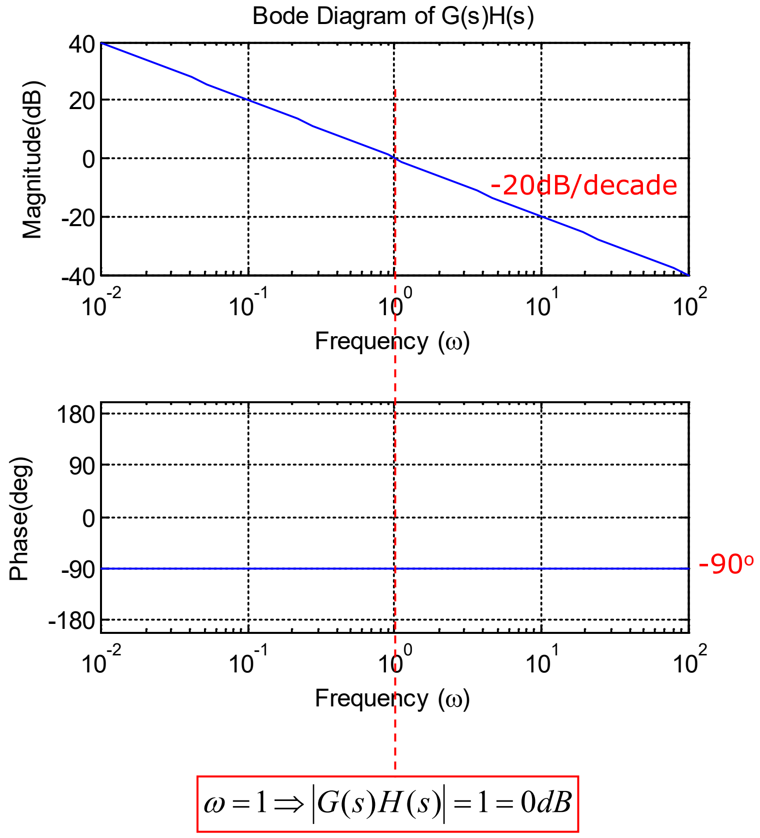
Typical Bode Plot, GH=1/s

Bode Plot of

$$G(s)H(s) = \frac{1}{s}$$

Substitute $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{1}{j\omega}$$



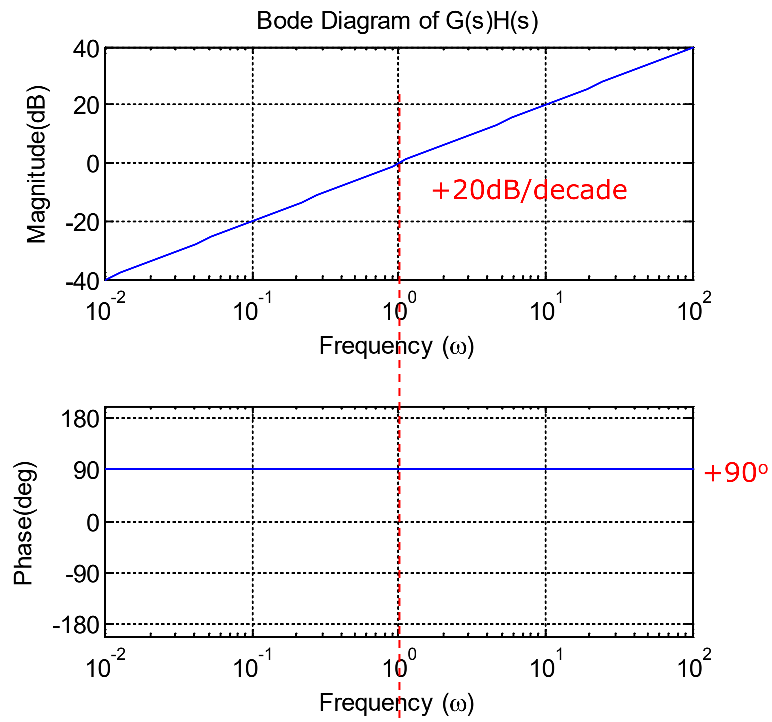
Typical Bode Plot, GH=s

Bode Plot of

$$G(s)H(s) = s$$

Substitute $s = j\omega$

$$G(j\omega)H(j\omega) = j\omega$$



$$\boxed{\omega = 1 \Rightarrow |G(s)H(s)| = 1 = 0dB}$$

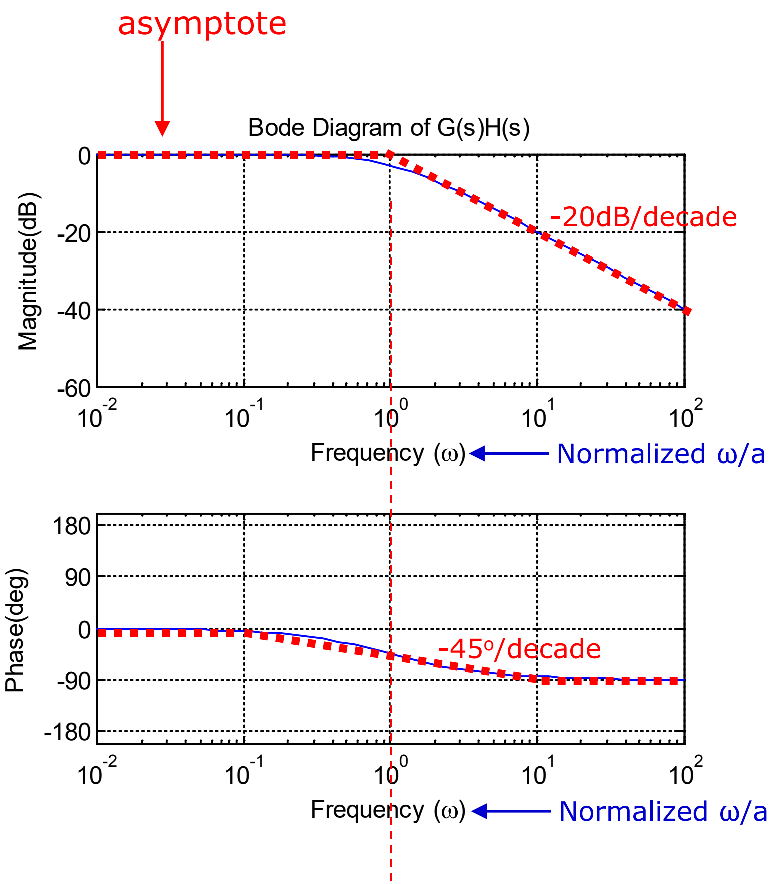
Typical Bode Plot, GH=a/(s+a)

Bode Plot of

$$G(s)H(s)=\frac{a}{s+a}$$

Substitute $s=j\omega$

$$G(j\omega)H(j\omega)=\frac{a}{j\omega+a}$$



$\omega=a$

$\Rightarrow |G(s)H(s)|=1=0dB$

$\Rightarrow \angle G(s)H(s)=45^\circ$

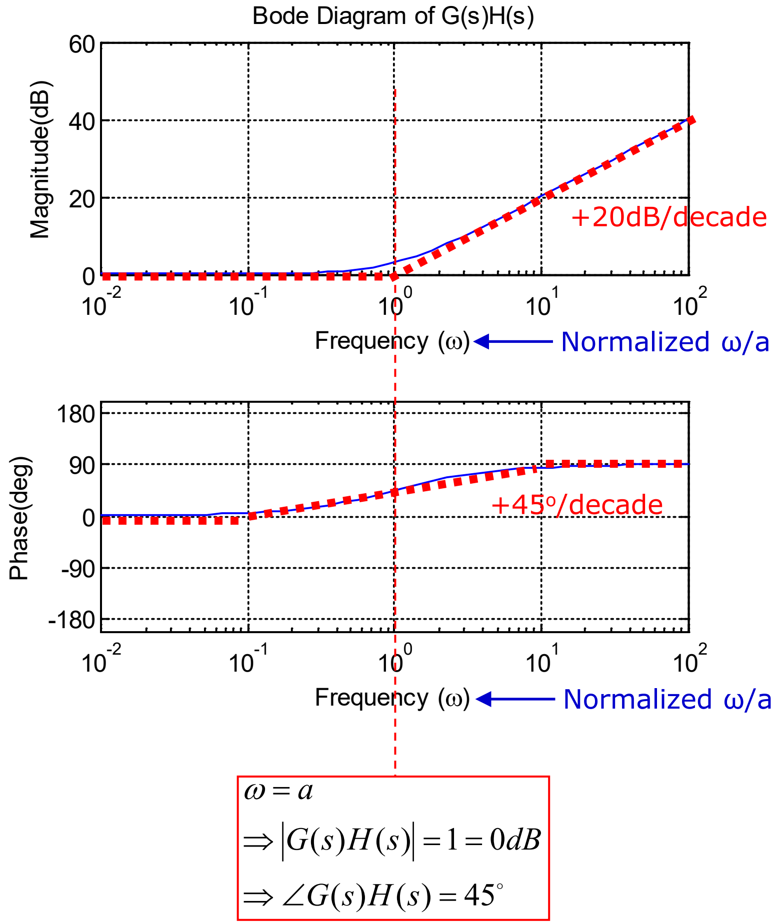
Typical Bode Plot, GH=(s+a)/a

Bode Plot of

$$G(s)H(s)=\frac{s+a}{a}$$

Substitute $s = j\omega$

$$G(j\omega)H(j\omega)=\frac{j\omega+a}{a}$$



Typical Bode Plot, Second-order

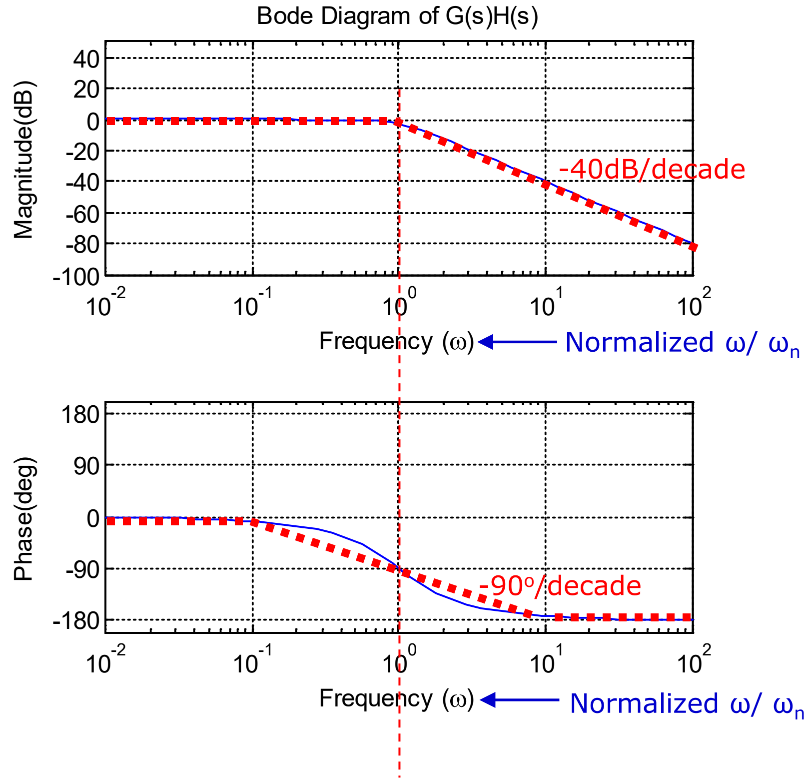
Bode Plot of

$$G(s)H(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

Substitute $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{\omega_n^{\ 2}}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^{\ 2}}$$

This example: $\omega_n=1$, $\zeta=0.7$



$$\omega = \omega_n$$
$$\Rightarrow \angle G(s)H(s) = 90^\circ$$

Example 1: Construct of Bode Plot

Plot the Bode Plot of $G(s)H(s)$

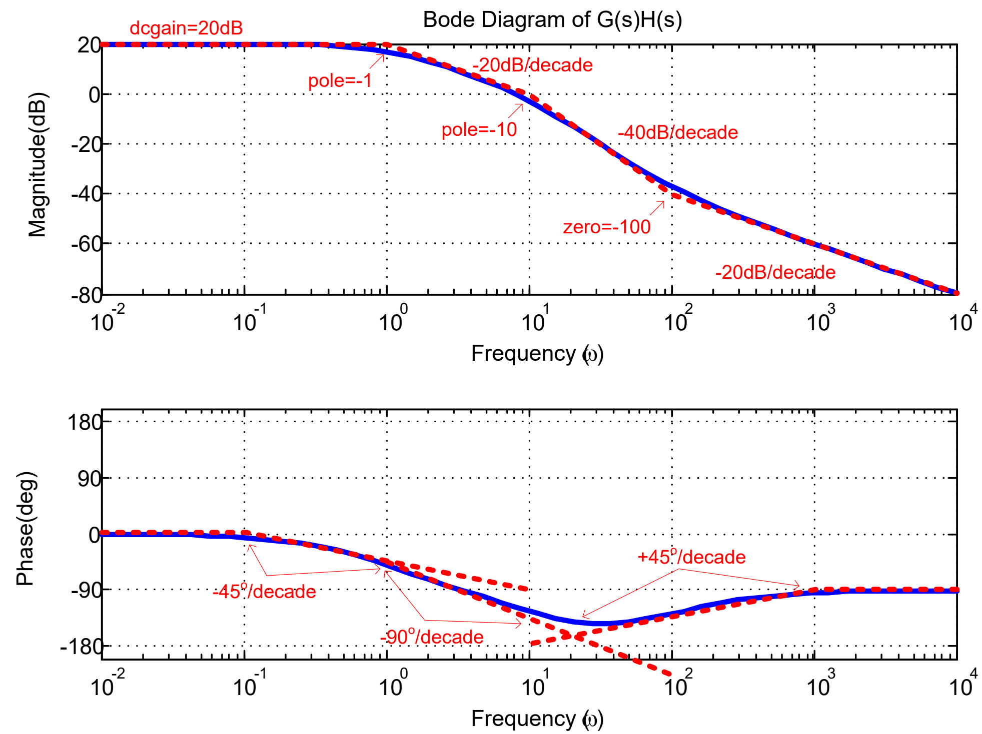
$$G(s)H(s) = \frac{s+100}{(s+1)(s+10)}$$

Re-write the $G(s)H(s)$ into standard form

$$G(s)H(s) = \frac{100}{1\cdot 10}\,\frac{1}{s+1}\,\frac{10}{s+10}\,\frac{s+100}{100}$$

$$G(s)H(s) = 10\cdot\frac{1}{s+1}\,\frac{10}{s+10}\,\frac{s+100}{100}$$

dcgain = 10
dcgain(dB) = 20*log(dcgain)=20dB



Example 2: Construct of Bode Plot

Plot the Bode Plot of $G(s)H(s)$

$$G(s)H(s) = \frac{100000}{s\left(s^2 + 100s + 5000\right)}$$

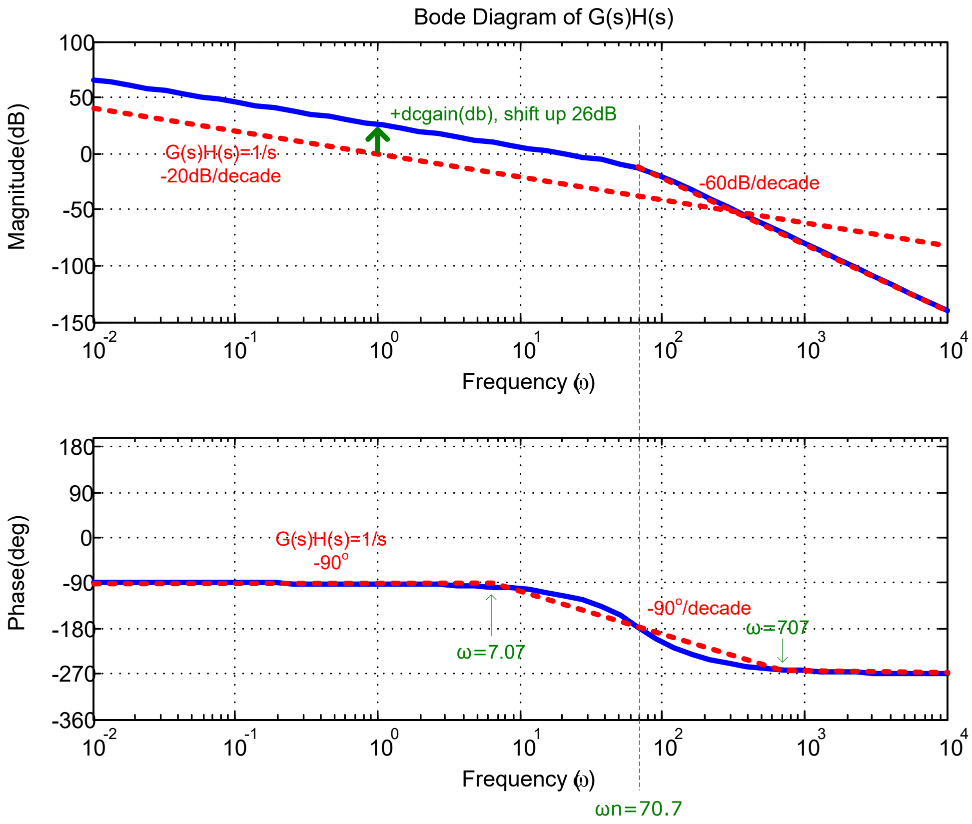
Re-write the $G(s)H(s)$ into standard form

$$G(s)H(s) = \frac{100000}{\left(\sqrt{5000}\right)^2} \frac{1}{s} \frac{\left(\sqrt{5000}\right)^2}{s^2 + 2 \cdot \frac{50}{\sqrt{5000}} \sqrt{5000}s + \left(\sqrt{5000}\right)^2}$$

$$G(s)H(s) = 20 \cdot \frac{1}{s} \cdot \frac{\left(\sqrt{5000}\right)^2}{s^2 + 2 \times 0.707 \times \sqrt{5000}s + \left(\sqrt{5000}\right)^2}$$

where $\omega_n = \sqrt{5000} = 70.7$ and $\zeta = 0.707$

dcgain = 20
dcgain(dB) = 20*log(dcgain)=26dB



Frequency Response Method - Nyquist Plot

Kelvin Leung, Ph.D.

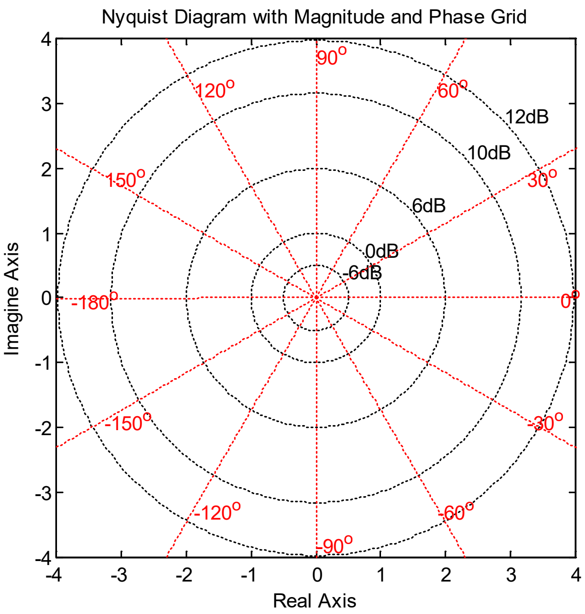
10-22-2012

http://www.kskelvin.net

Nyquist Plot

- Introduction
  - Bode plot and Nyquist plot are commonly used in the frequency-response representation of LTI (Linear Time Invariant) feedback control systems.
    - Bode plot is rectangular plot
    - Nyquist plot is polar plot
      - Includes the loci for both ω>0 and ω<0.

Black Line: Magnitude of G(s)H(s)
Red Line: Phase of G(s)H(s)



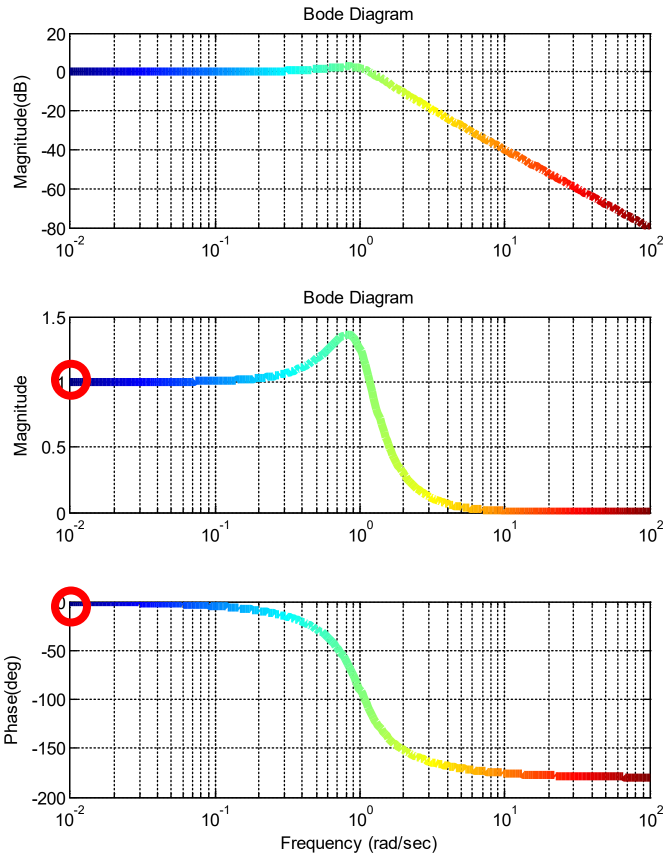
$$Mag_{dB} = 20\log_{10}(Mag)$$

or

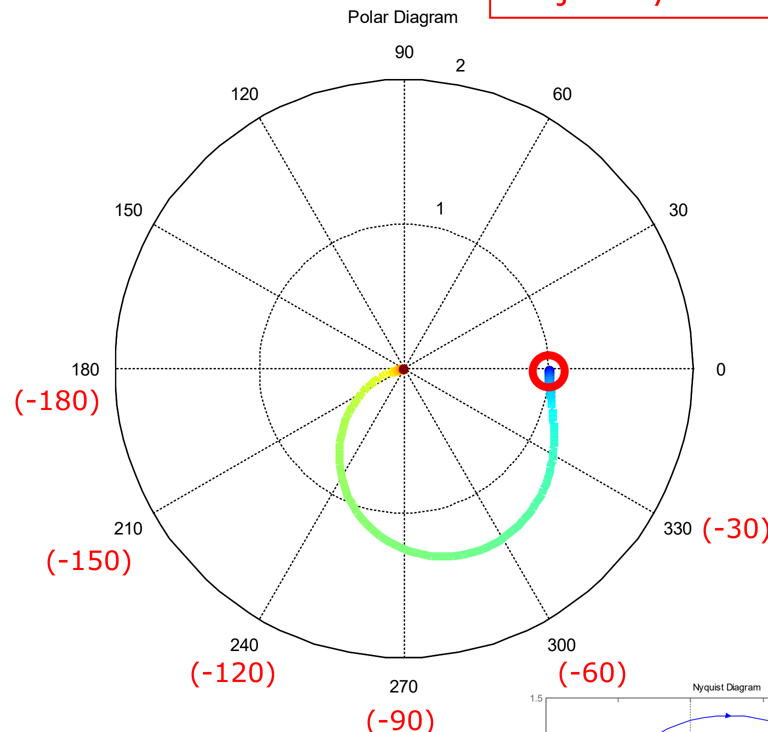
$$Mag = 10^{\frac{Mag_{dB}}{20}}$$

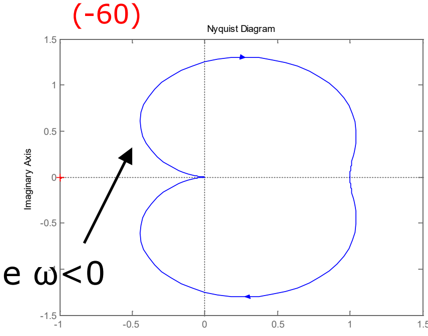
nyquist_polar_grid.m

Bode and Polar plots



Trajectory has direction!



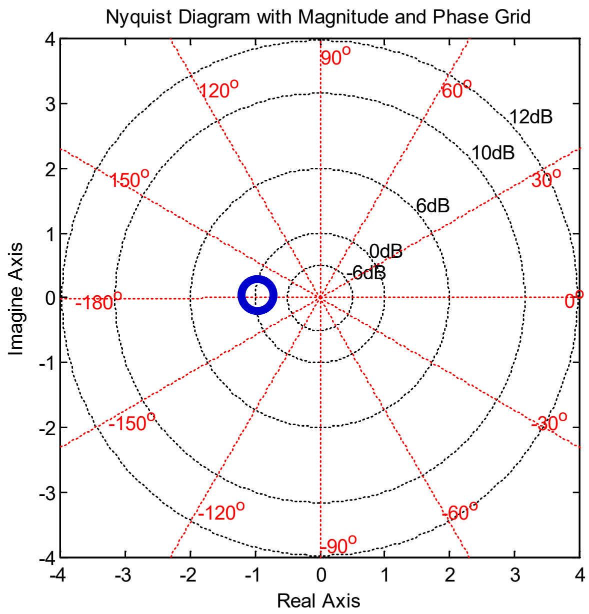


Nyquist plot include ω<0

bode_nyquist_polar.m

Special Point in Nyquist Plot

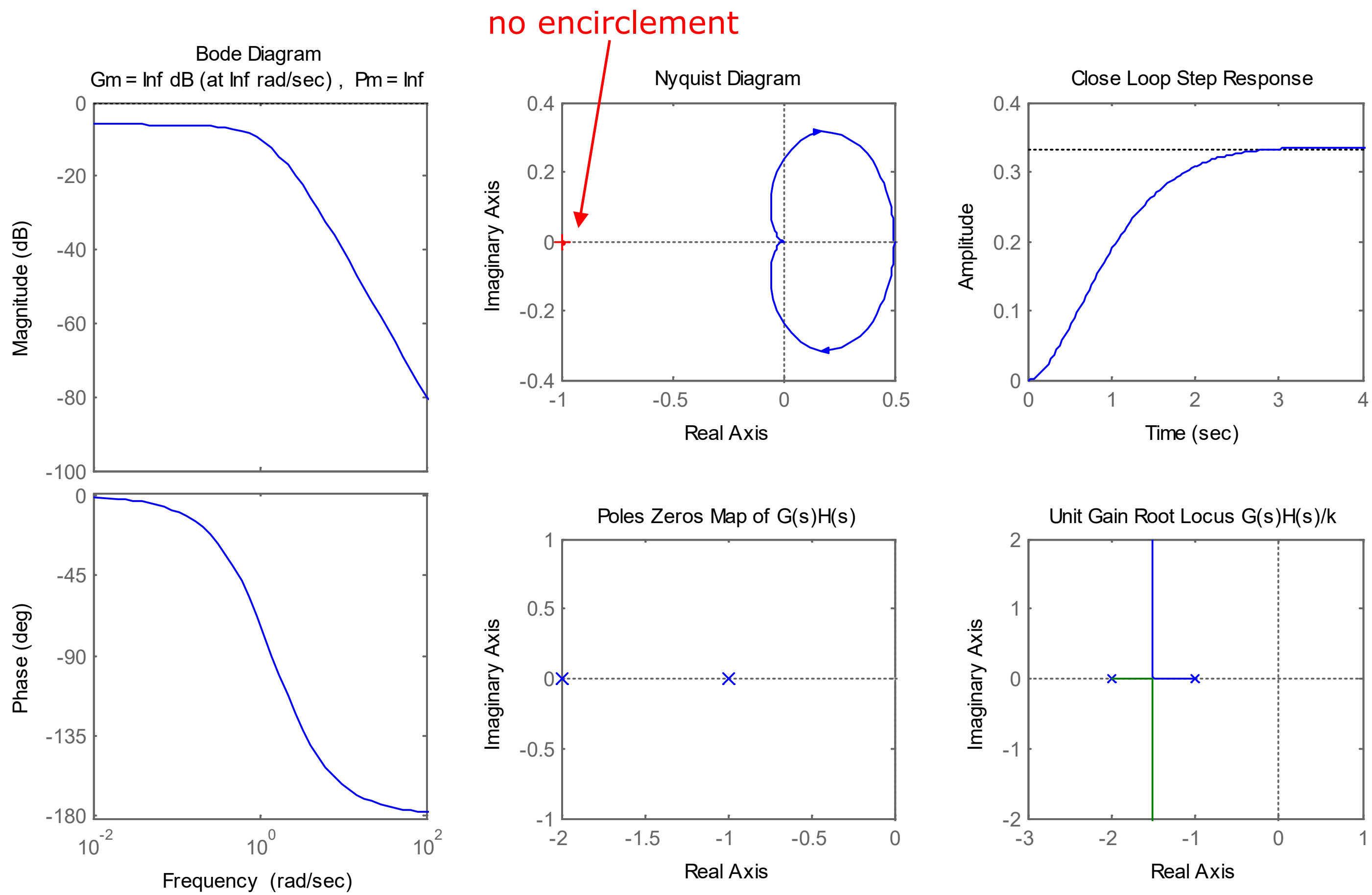
- Point: -1+j0
  - Magnitude = 1 = 0dB
  - Phase = -180°
  - In Bode plot, for a stable system
    - Condition 1
      - If G(s)H(s) doesn't have right half plane poles
    - Condition 2
      - |G(s)H(s)| <0dB when Phase = -180o. Therefore, this point is critical in Nyquist plot.



Stability Analysis of Nyquist plot

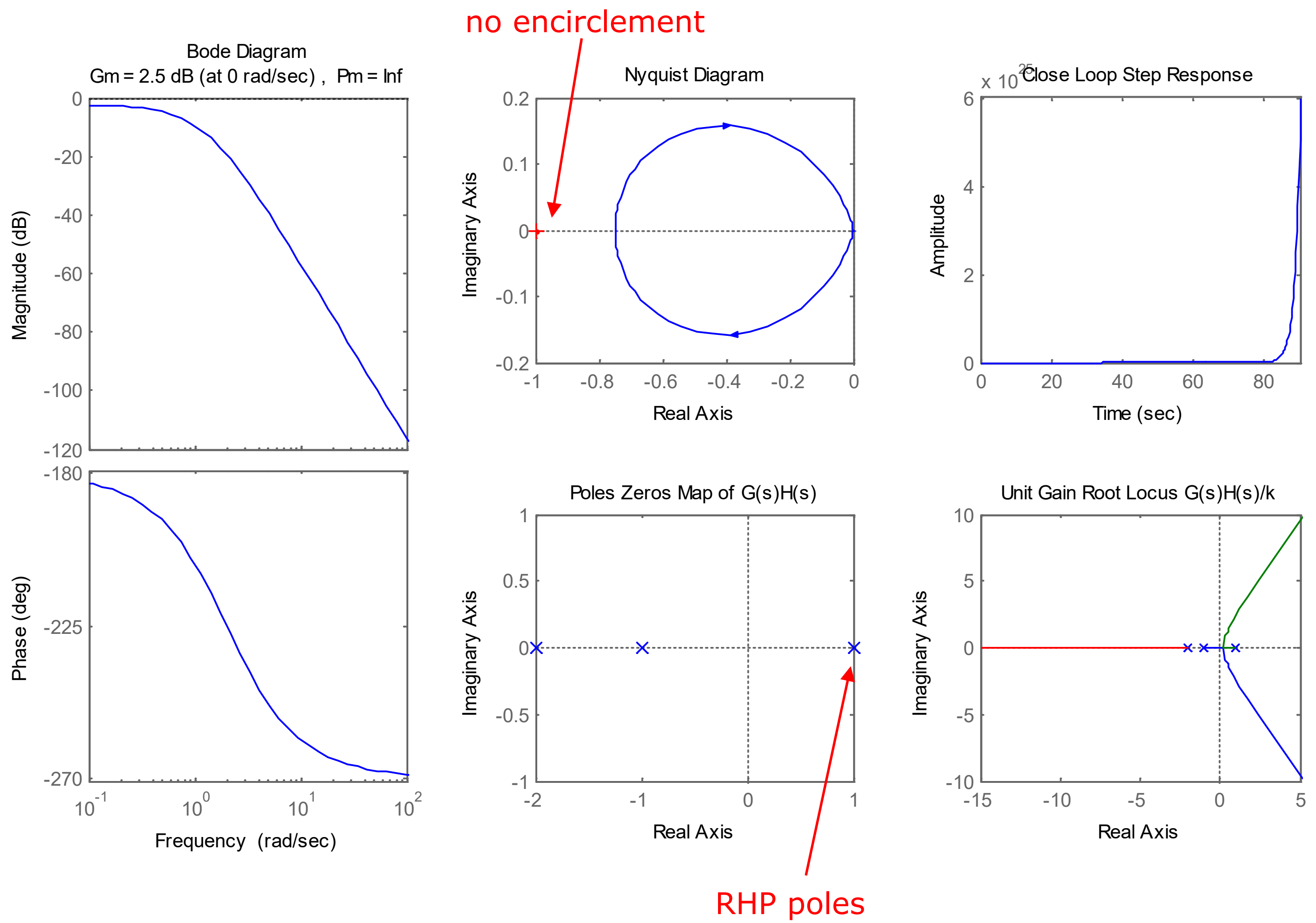
- Reference
  - Page 454, “Modern Control Engineering (5th Edition)”, Katsuhiko Ogata.
- Rules
  1. There is no encirclement of the -1+j0 point. This implies that the system is stable if there are no poles of G(s)H(s) in the right-half s plane; otherwise, the system is unstable.
  2. There are one or more counterclockwise encirclements of the -1+j0 point. In this case the system is stable if the number of counterclockwise encirclements is the same as the number of poles of G(s)H(s) in the right-half s plane; otherwise, the system is unstable.
  3. There are one or more clockwise encirclements of the -1+j0 point. In this case the system is unstable.

Rule #1, Stable System (no encirclement of -1+j0)



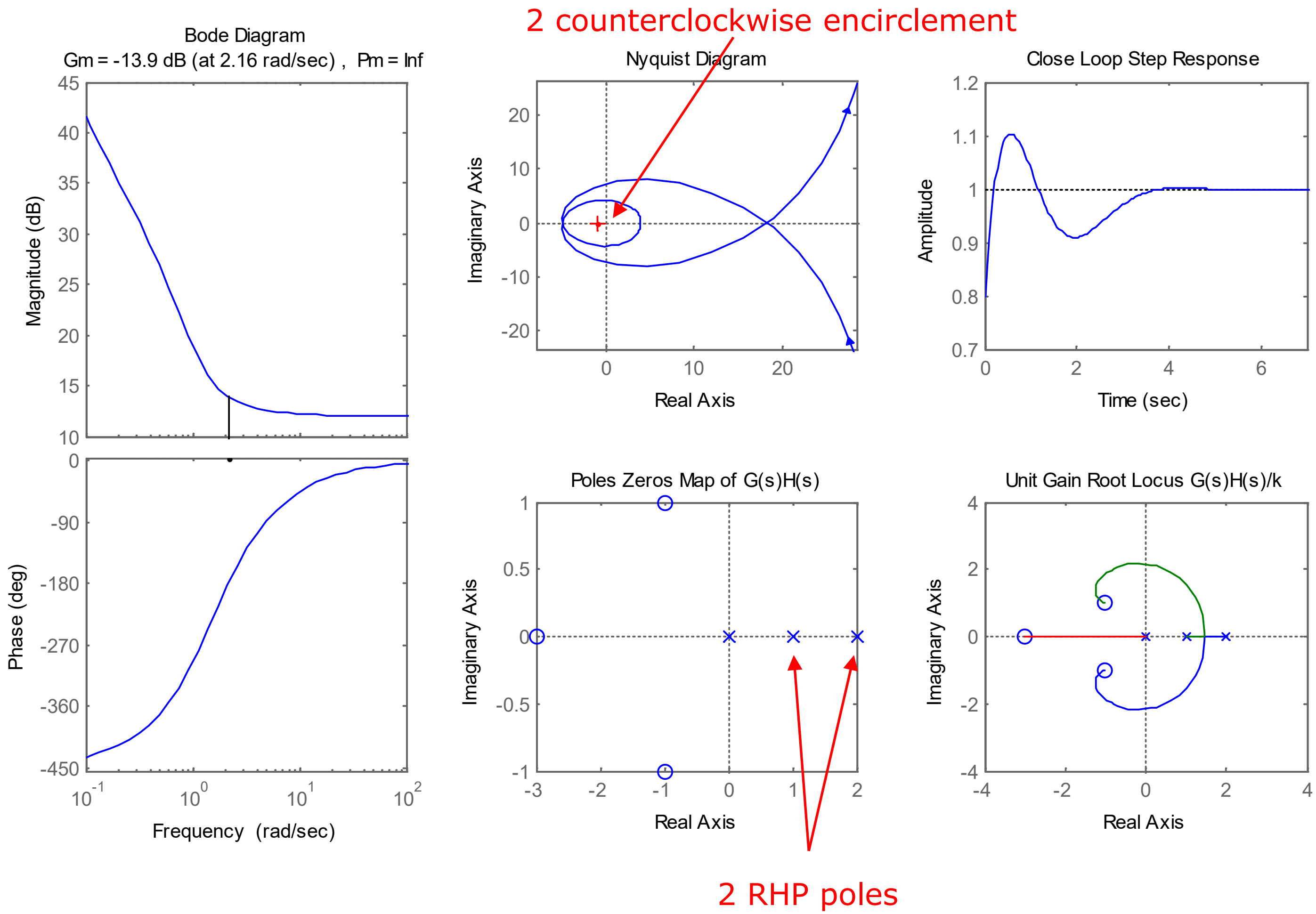
nyquist_stability_rule.m

Rule #1, Unstable System (no encirclement of -1+j0)



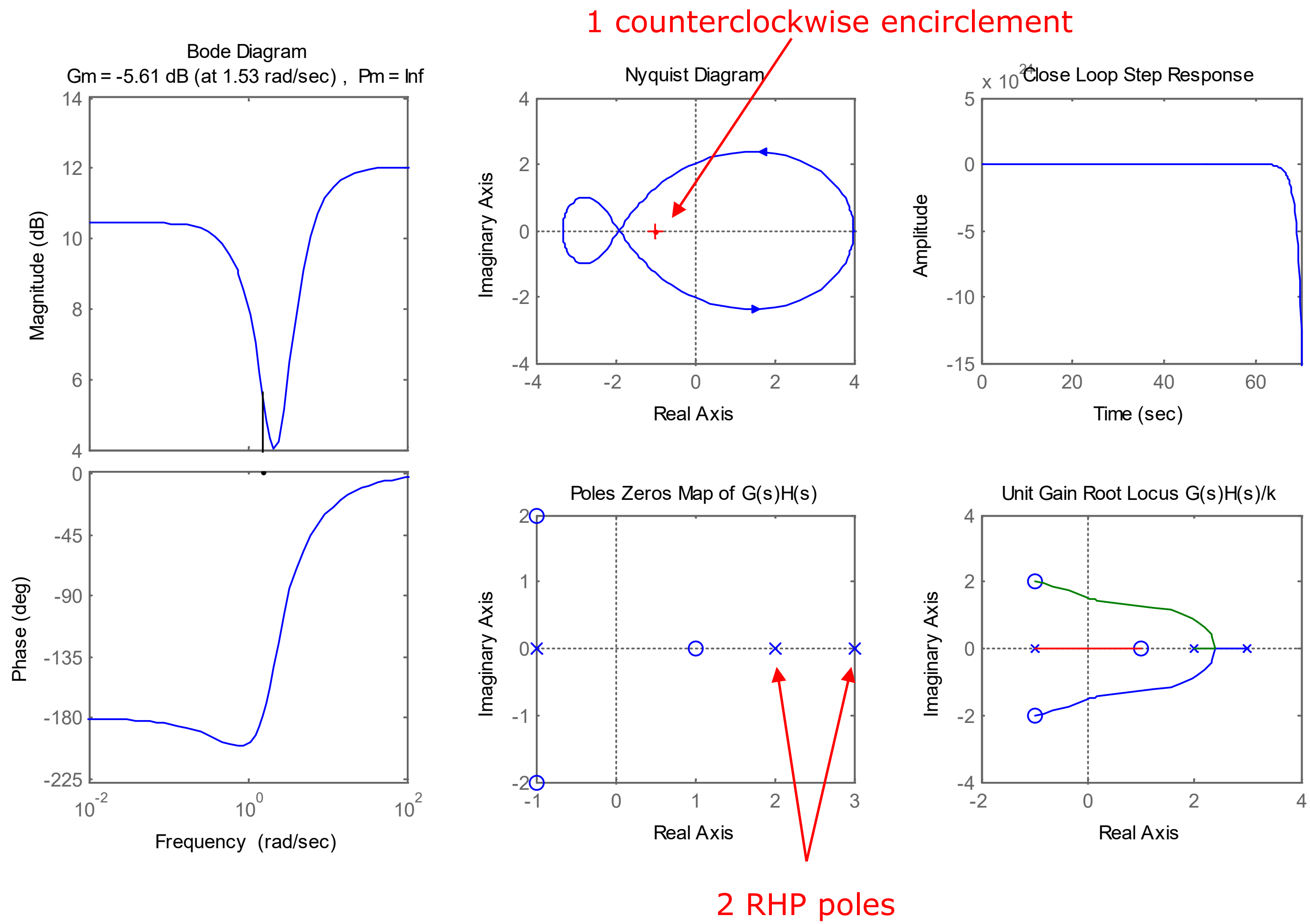
nyquist_stability_rule.m

Rule #2, Stable System (counterclockwise encirclement of -1+j0)



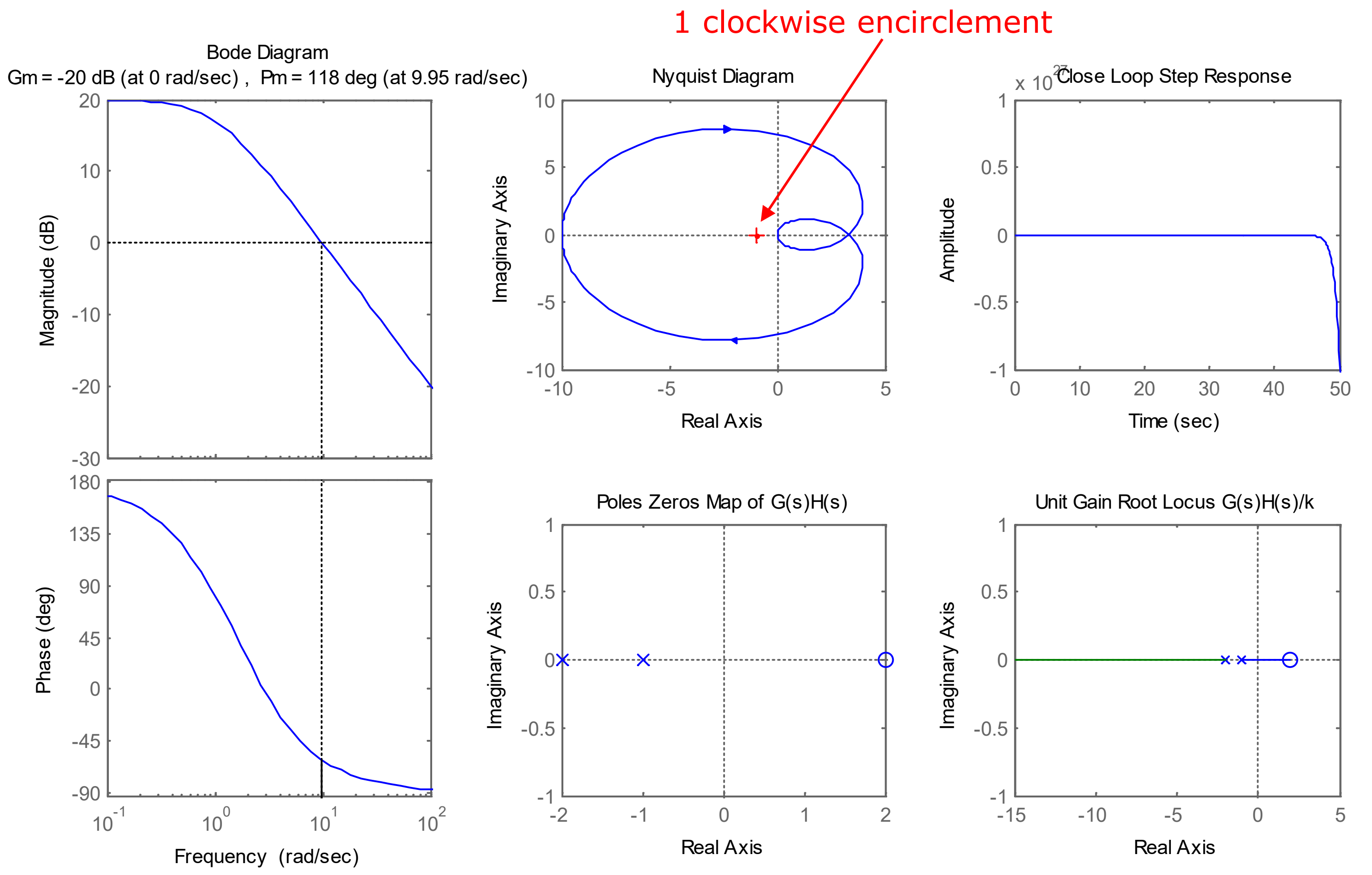
nyquist_stability_rule.m

Rule #2, Unstable System (counterclockwise encirclement of -1+j0)



nyquist_stability_rule.m

Rule #3, Unstable System (clockwise encirclement of -1+j0)

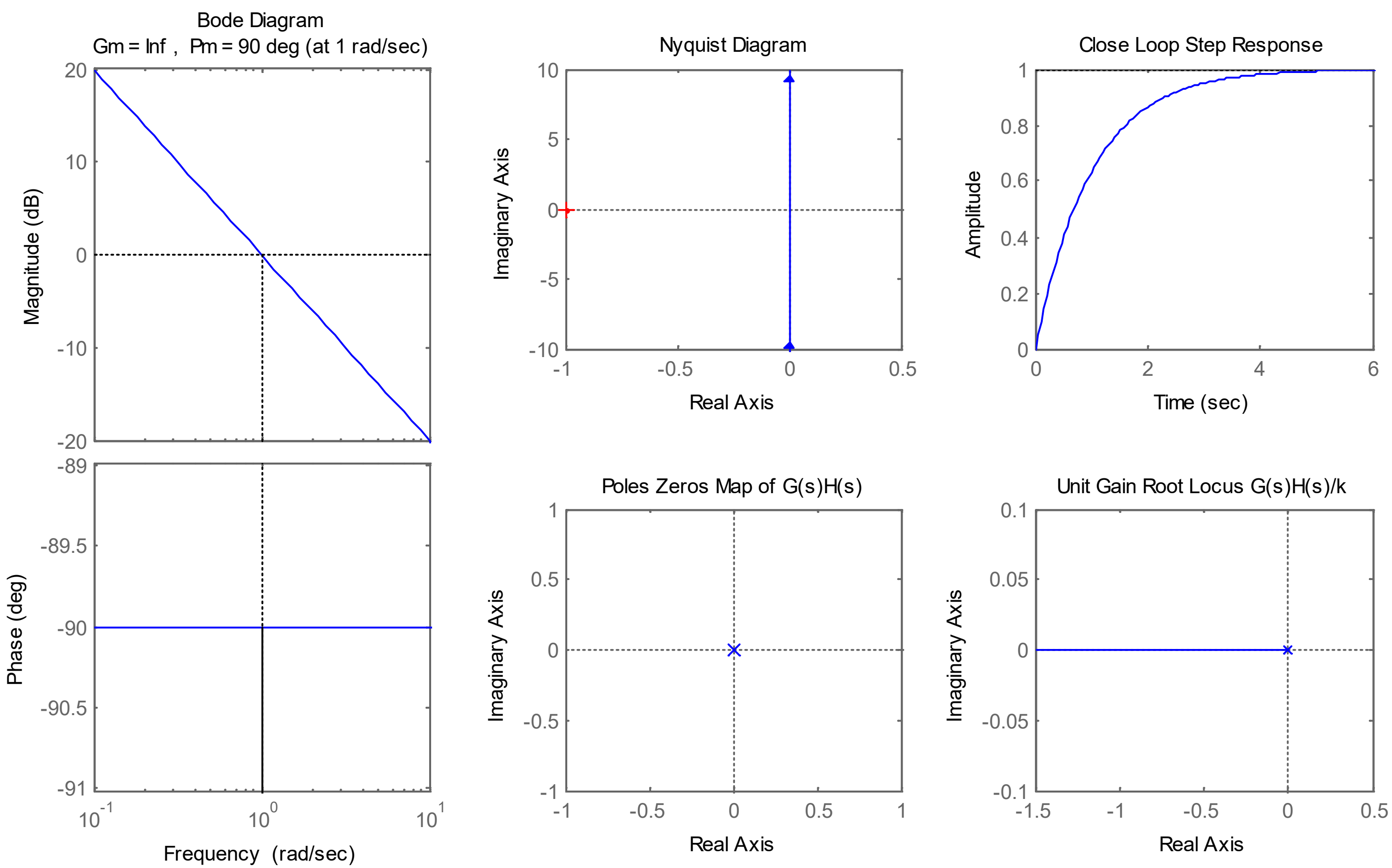


nyquist_stability_rule.m

Appendix

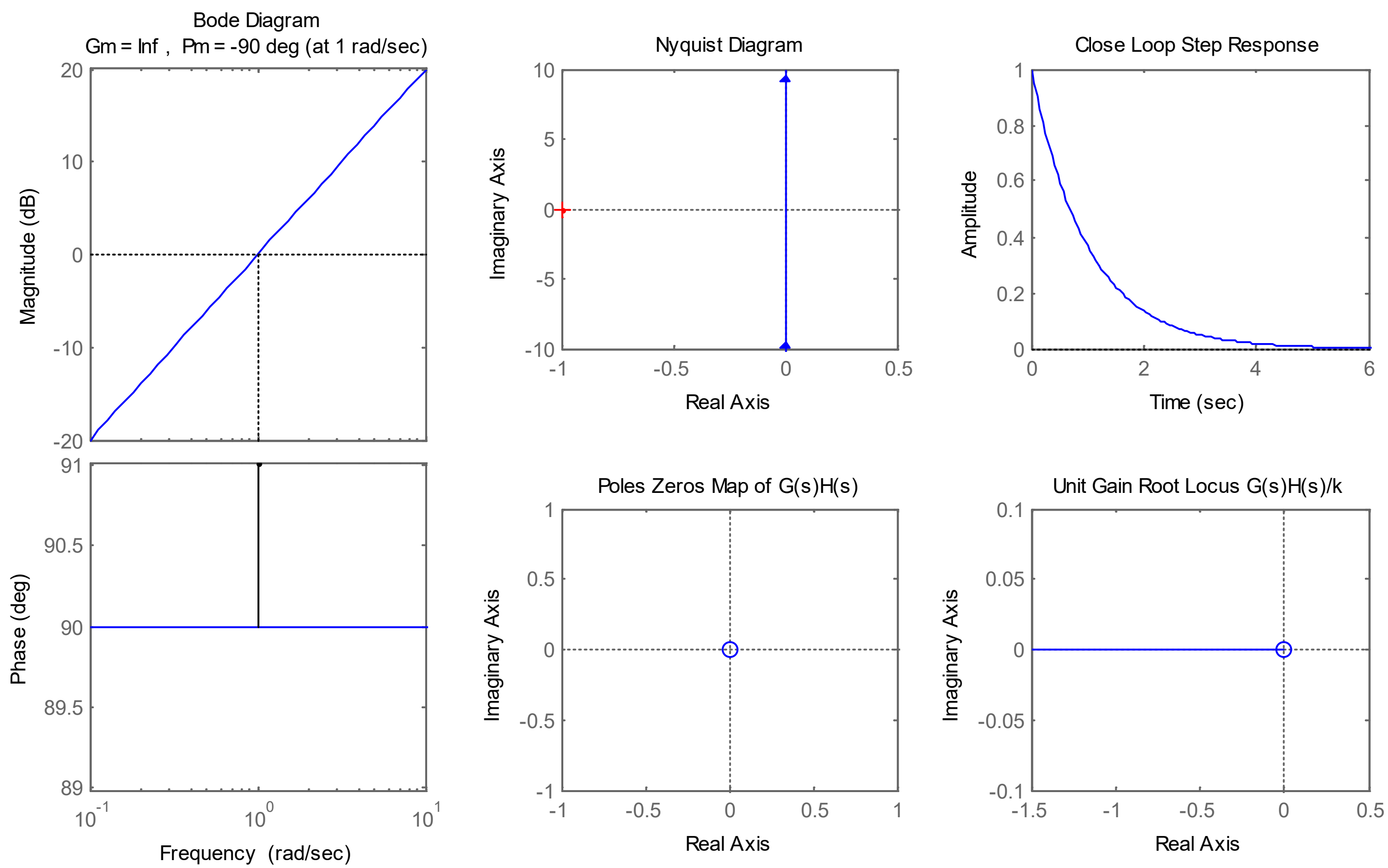
Standard G(s)H(s) Bode and Nyquist Plots

$$G(s)H(s) = \frac{1}{s}$$



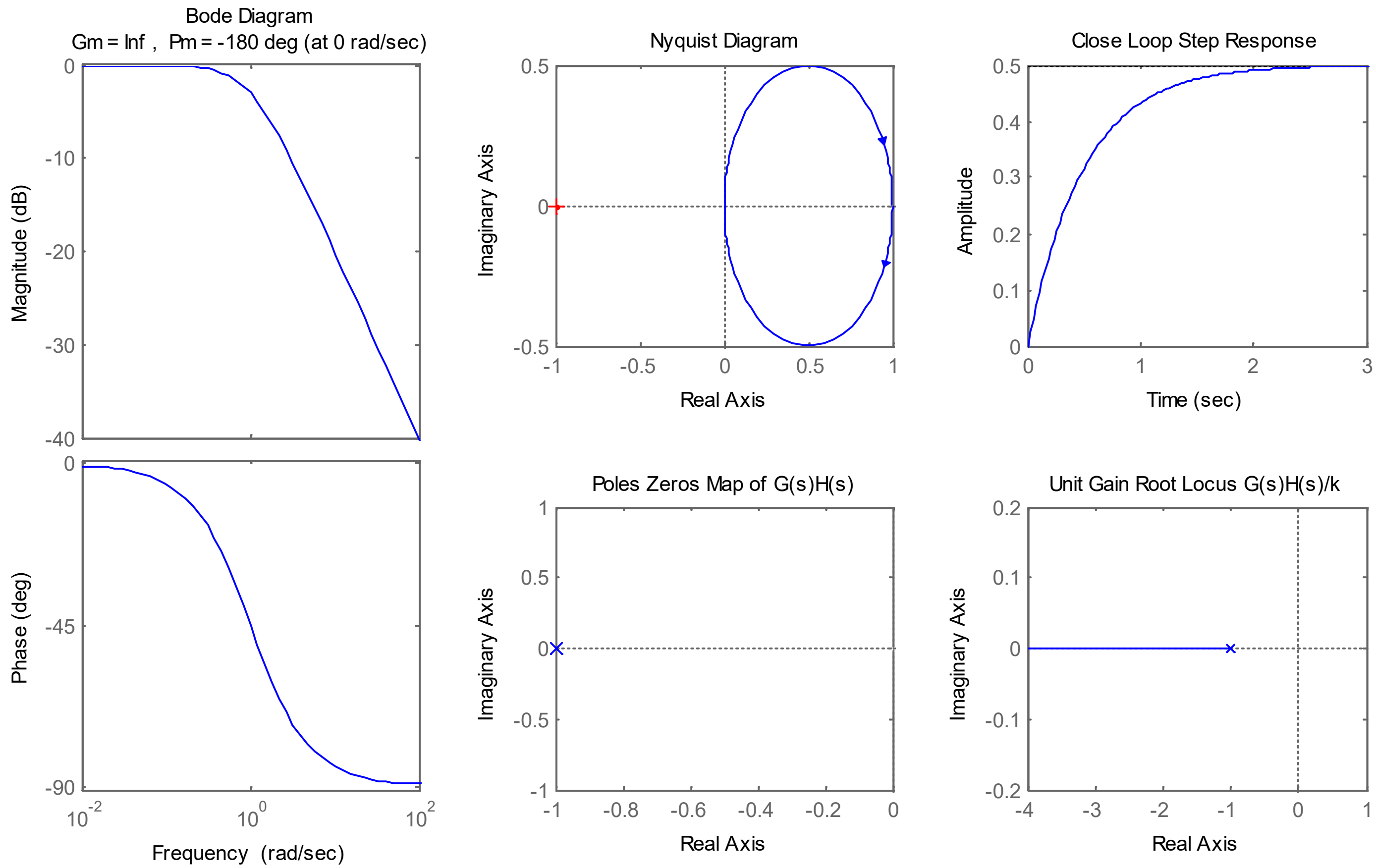
Bode Diagram
Gm = Inf , Pm = 90 deg (at 1 rad/sec)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
Nyquist Diagram
Imaginary Axis
Real Axis
Close Loop Step Response
Amplitude
Time (sec)
Poles Zeros Map of G(s)H(s)
Imaginary Axis
Real Axis
Unit Gain Root Locus G(s)H(s)/k
Imaginary Axis
Real Axis

$$G(s)H(s) = s$$

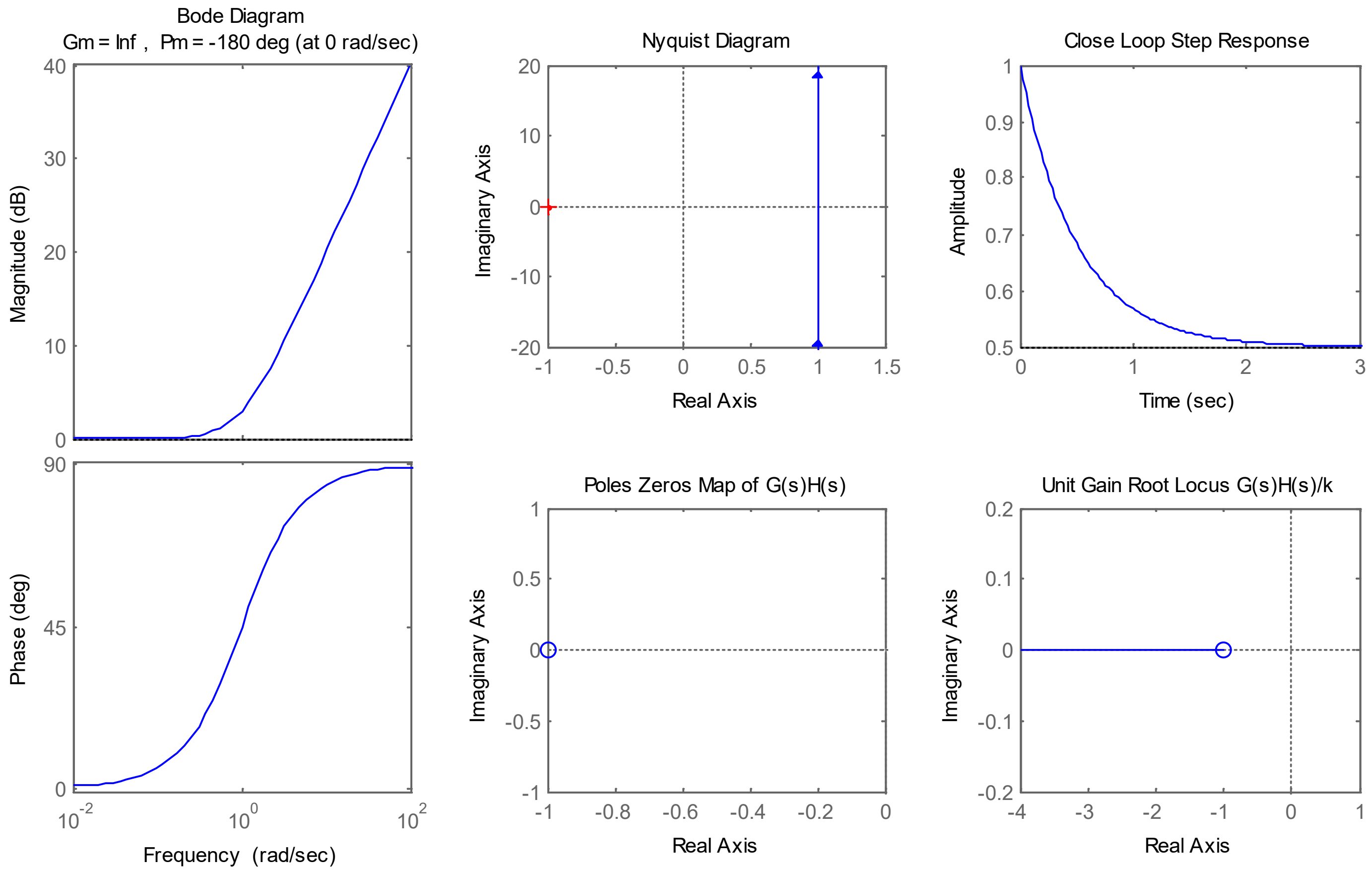


Bode Diagram
Gm = Inf , Pm = -90 deg (at 1 rad/sec)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
Nyquist Diagram
Imaginary Axis
Real Axis
Close Loop Step Response
Amplitude
Time (sec)
Poles Zeros Map of G(s)H(s)
Imaginary Axis
Real Axis
Unit Gain Root Locus G(s)H(s)/k
Imaginary Axis
Real Axis

$$G(s)H(s) = \frac{1}{s+1}$$

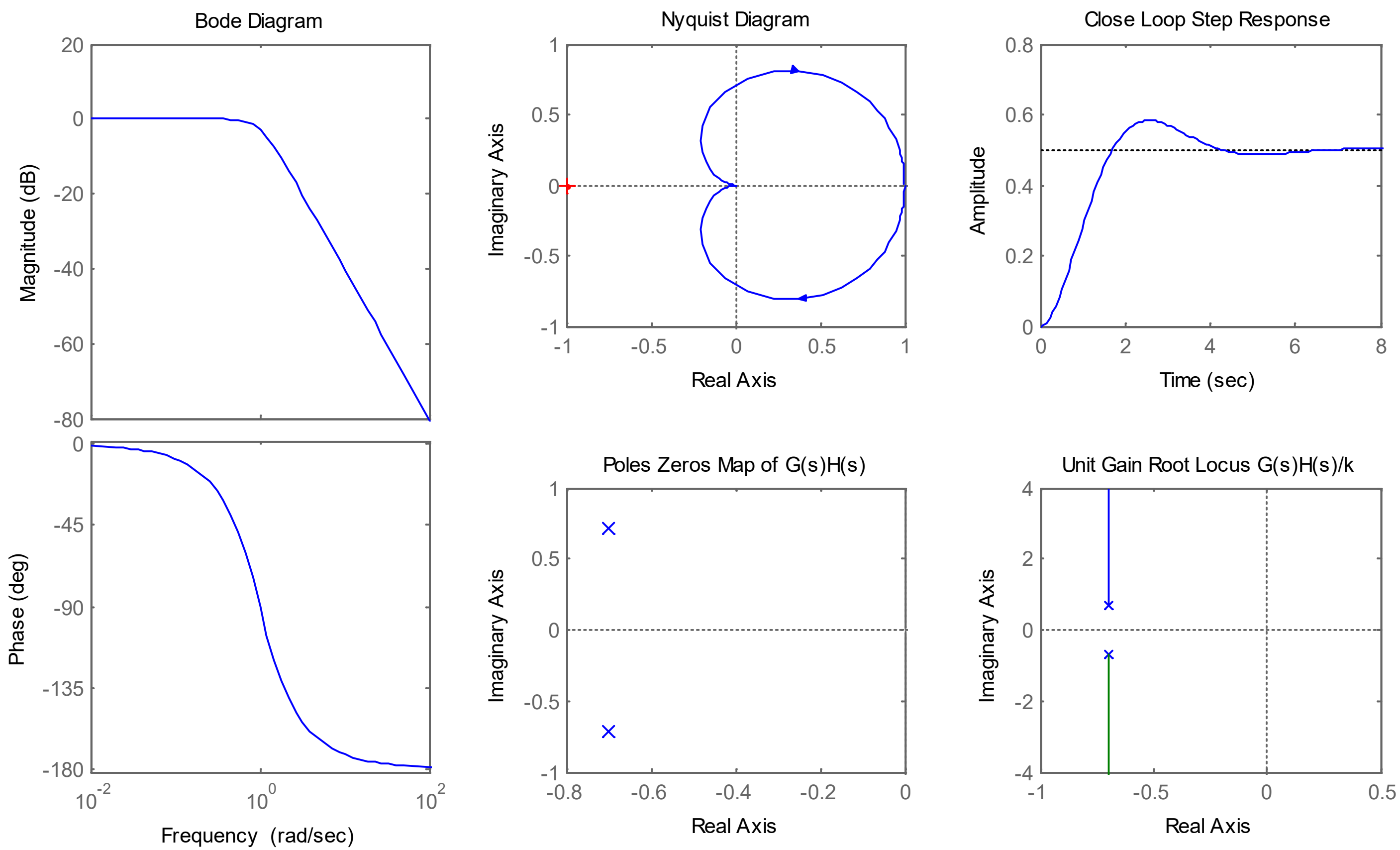


$$G(s)H(s) = s + 1$$



Bode Diagram
Gm = Inf , Pm = -180 deg (at 0 rad/sec)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
Nyquist Diagram
Imaginary Axis
Real Axis
Close Loop Step Response
Amplitude
Time (sec)
Poles Zeros Map of G(s)H(s)
Imaginary Axis
Real Axis
Unit Gain Root Locus G(s)H(s)/k
Imaginary Axis
Real Axis

$$G(s)H(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}} \text{ where } \omega_n = 1, \zeta = 0.7$$



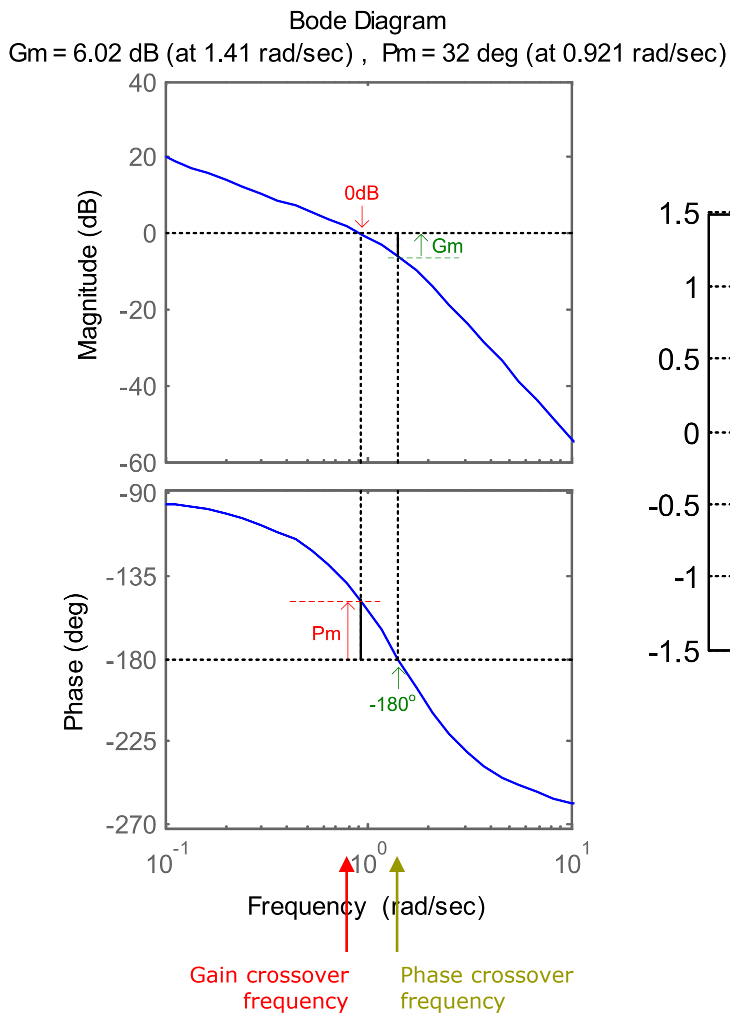
Gain and Phase Margins

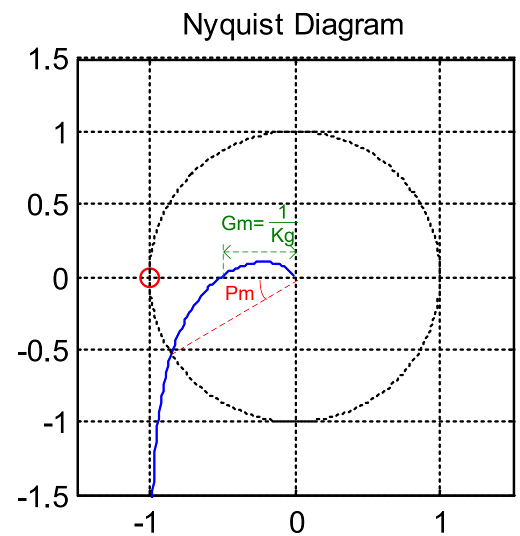
Kelvin Leung, Ph.D.

10-22-2012

http://www.kskelvin.net

Gain and Phase Margins Definition in Bode and Nyquist plots (stable)





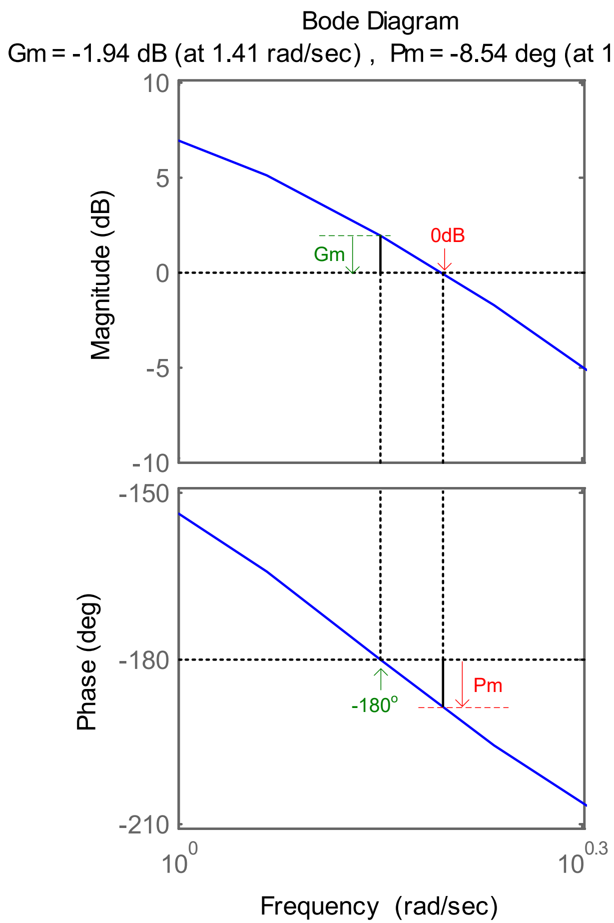
Numerial Example for Nyquist plot

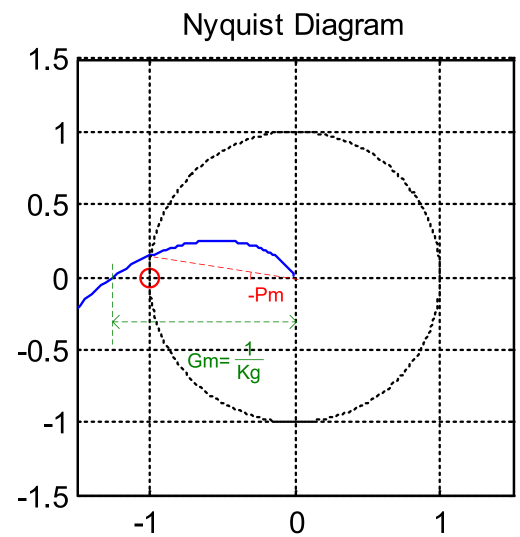
$$K_g = 0.5$$

$$Gm = \frac{1}{K_g} = 2 = 20\log(2) = 6.02dB$$

$$Pm = \tan^{-1}\left(\frac{0.5305}{0.8476}\right) = 32^\circ$$

Gain and Phase Margins Definition in Bode and Nyquist plots (unstable)





Numerial Example for Nyquist plot

$$K_g = 1.25$$

$$Gm = \frac{1}{K_g} = 0.8 = 20\log(0.8) = -1.94dB$$

$$Pm = -\tan^{-1}\left(\frac{0.1484}{0.9889}\right) = -8.54^{\circ}$$

Relationship between open-loop and close-loop response

- Natural frequency (ω_n)
  - ω_n in closed-loop system is somewhere between the gain crossover frequency and phase crossover frequency in open-loop system.
    - page. 473-474 of "Modern Control Engineering", Ogata, 5th Edition
  - A very rough estimate is that the bandwidth (freq @ -3dB) is approximately equal to the natural frequency.
    - [http://www.engin.umich.edu/class/ctms/freq/freq.htm]
- Damping ratio (ζ)
  - Phase margin in open-loop system has linear relationship with ζ of closed-loop system
    - Exact Formula

      Phase margin (γ) and Damping Factor (ζ)

      $$\gamma = \tan^{-1} \frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}$$

    - Approximation: for $\zeta < 0.6$, $\zeta = 0.01$ Pm

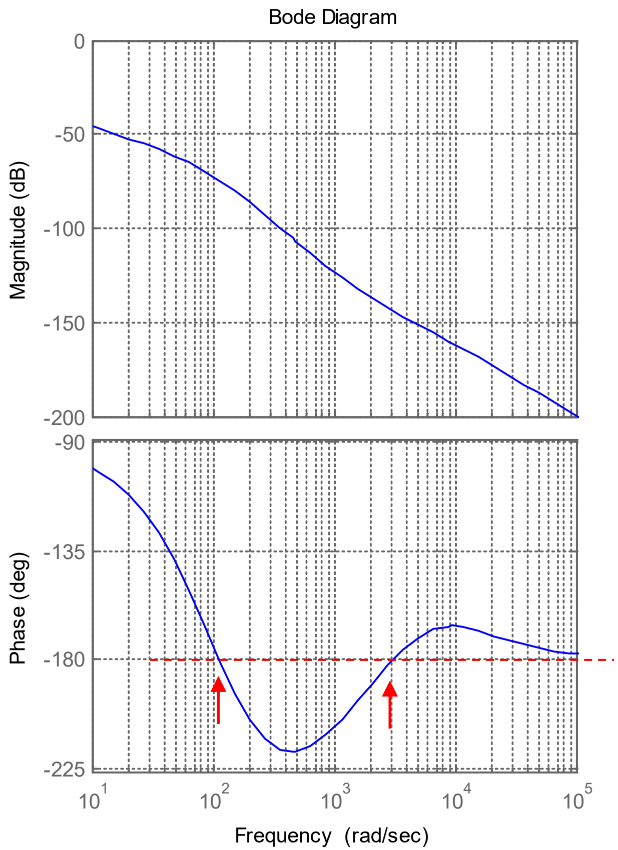
Design with Bode Plot

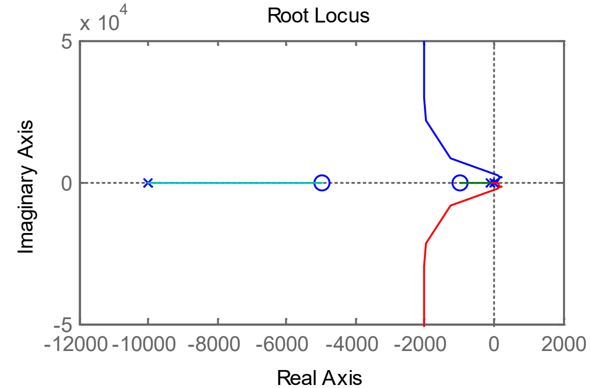
- Design criteria
  - Bode plot can only be used to design close-loop feedback from stable open-loop system (i.e. G(s)H(s) doesn't has RHP poles), otherwise, Nyquist or Root-Locus need to be used.
    - Reason: Refer to stability rule #1 in Nyquist plot powerpoint.
  - System performance
    - DC Gain
      - Determine the steady state error
        - Increase of DC Gain, Decrease of Steady State Error
    - Phase margin
      - Determine the damping ratio and overshoot.
        - Phase margin is normally selected to between 30^o-60^o.
    - Gain margin
      - Determine the robustness of system. Normally > 6dB.
        - To guarantee stability even if the open-loop gain and time constants of the components vary to a certain extend.
    - Gain/Phase crossover frequency
      - Determine the transient response speed.
        - Increase the crossover frequency, Increase transient speed.

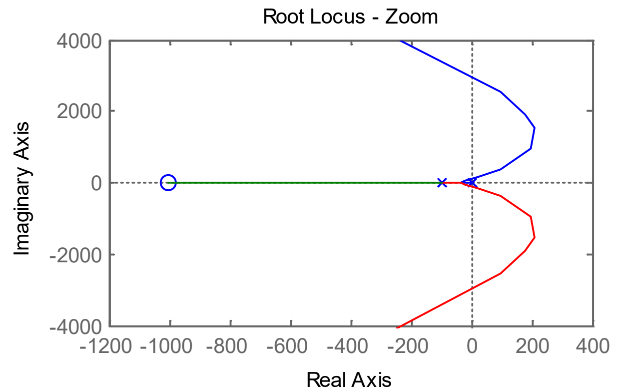
Stability of multiple phase crossover frequencies system

Stability of multiple phase crossover

Bode plot shows a system which has multiple phase crossover at 180°

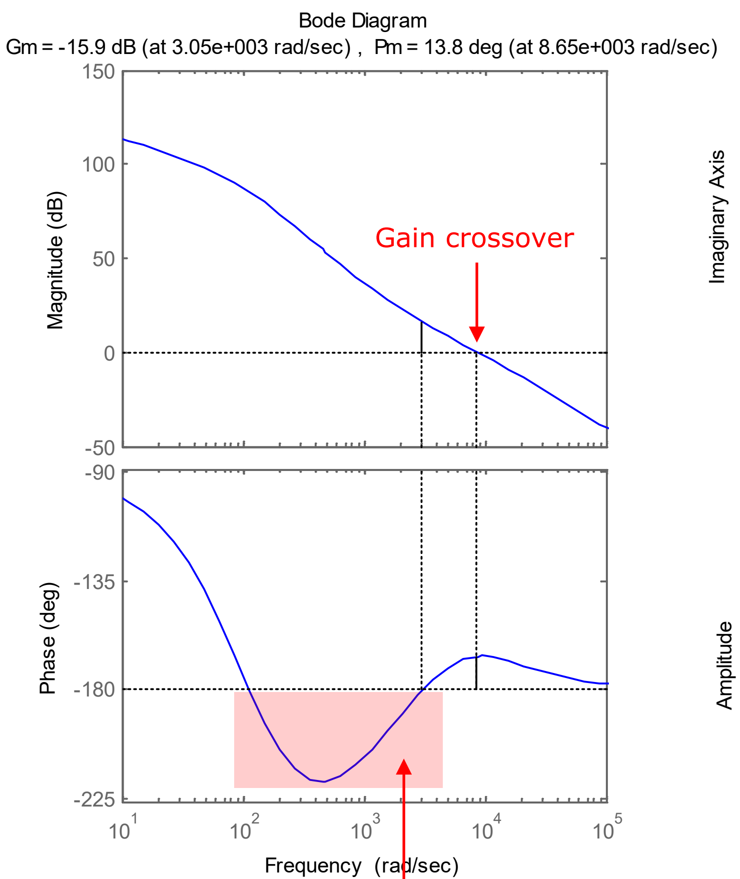


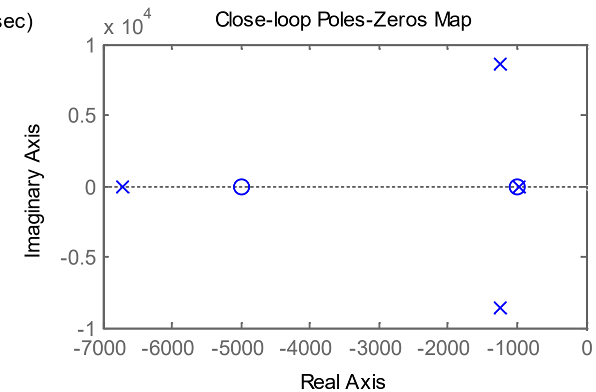


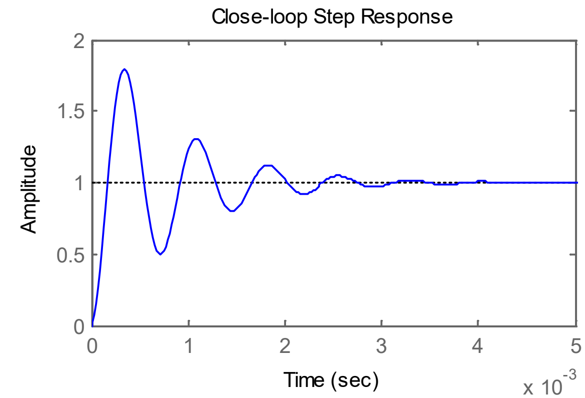


Root locus shows that there are 2 region of gain K which can give stable system
Therefore, root locus actually indicate “second” phase crossover can be used to generate a stable system

Stability of multiple phase crossover



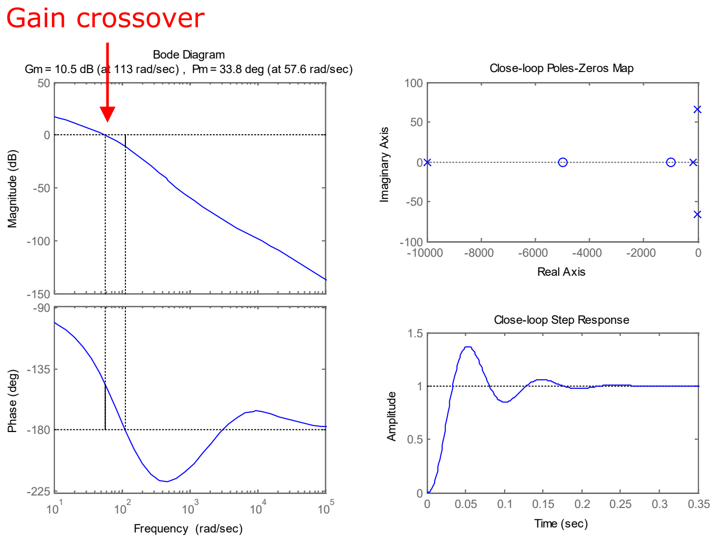




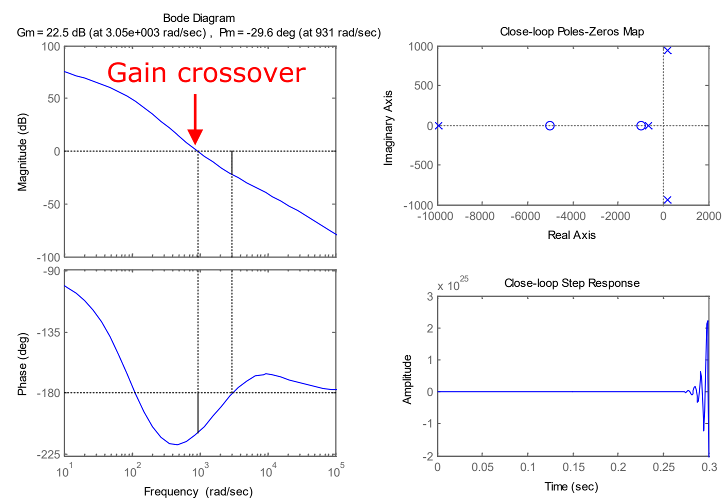
System is stable as long as phase margin is +ve, even the phase drops below -180° before that.

Stability of multiple phase crossover

Stable Case: Low Gain

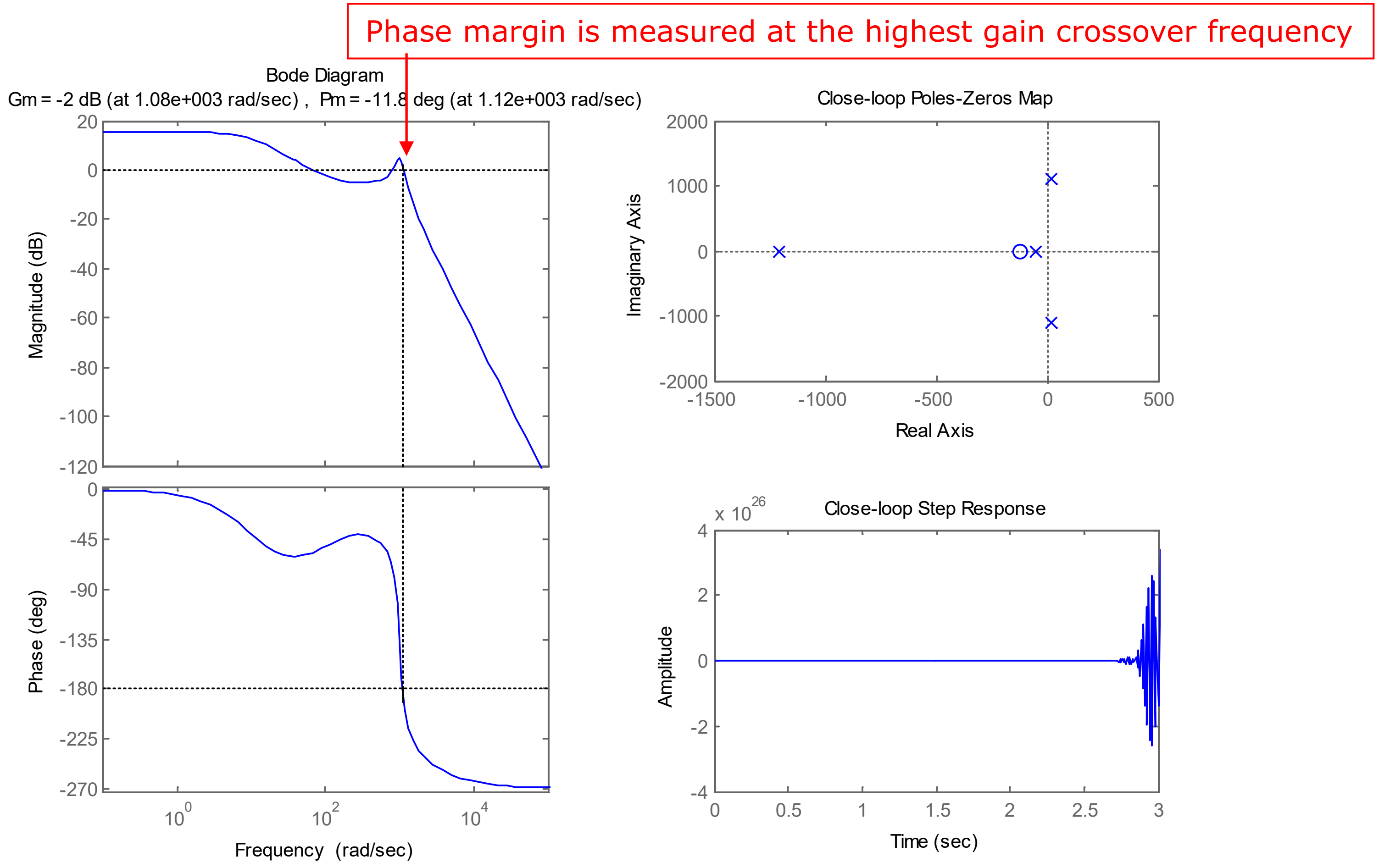


Unstable Case: Middle Gain



Stability of multiple gain crossover frequencies system

Stability of multiple gain crossover



Design with Lead or Lag Compensator

Kelvin Leung, Ph.D.

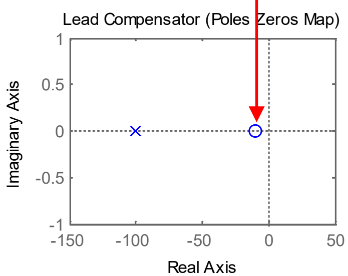
10-19-2012

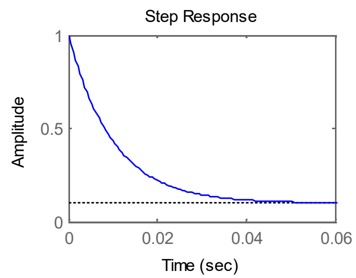
http://www.kskelvin.net

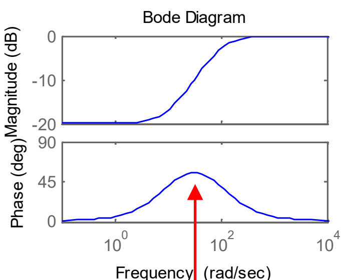
Lead and Lag Compensator Definitation

Lead Compensator

zero dominate







Add +ve phase

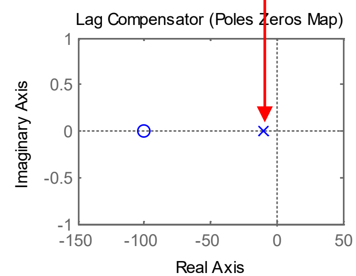
$$G_C(s) = \frac{s+z}{s+p}$$

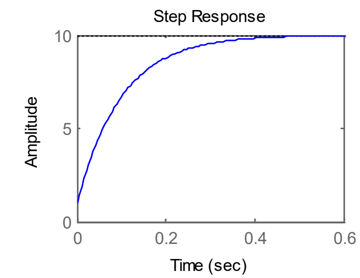
where $p > z$

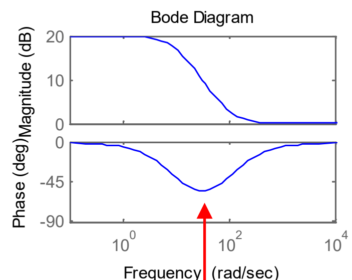
Application:
- Speed up system response

Lag Compensator

pole dominate







Subtract -ve phase

$$G_C(s) = \frac{s+z}{s+p}$$

where $z > p$

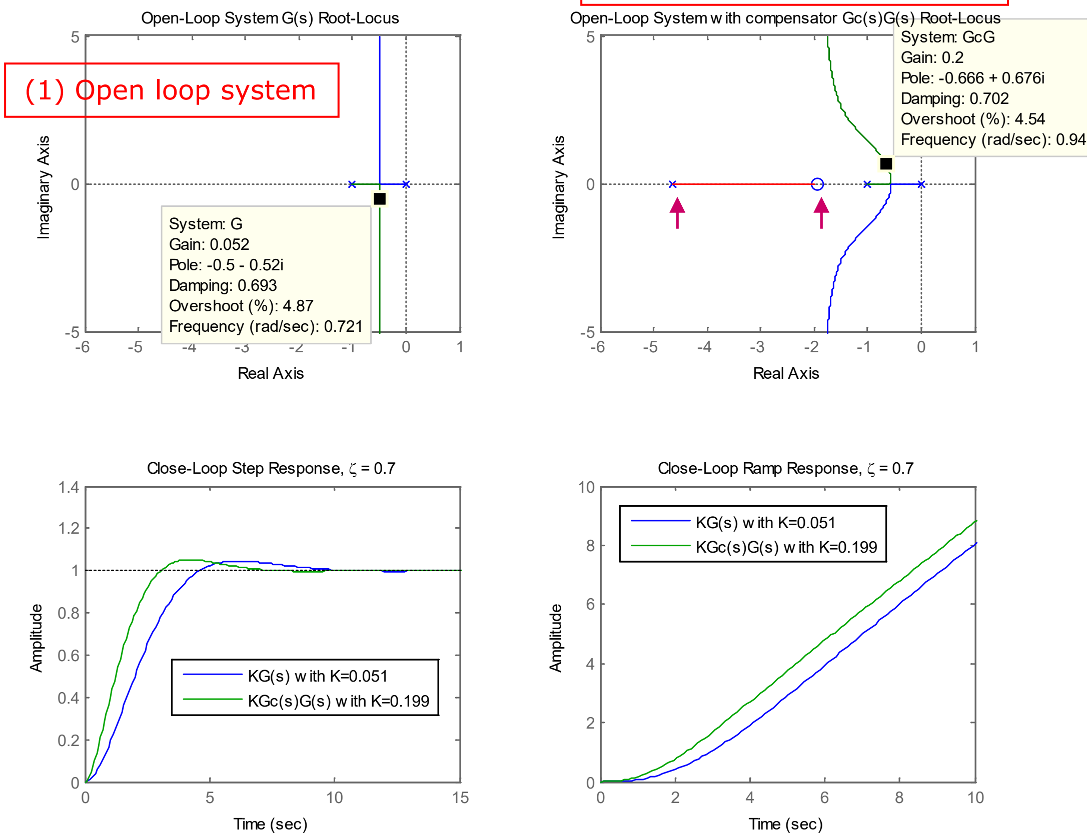
Application:
- Improve steady state error
- Avoid changing system dynamic

Design with lead compensator

Example of system compensation with lead compensator – root locus

(1) Open loop system

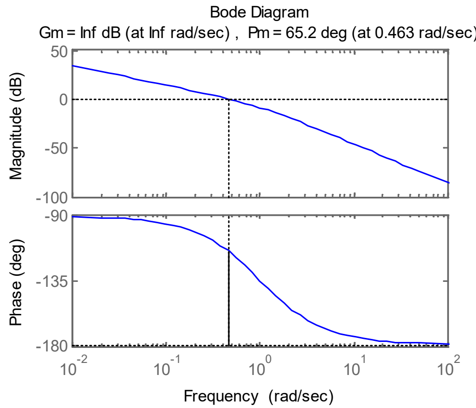
(2) Add a lead compensator to shift the root locus to left

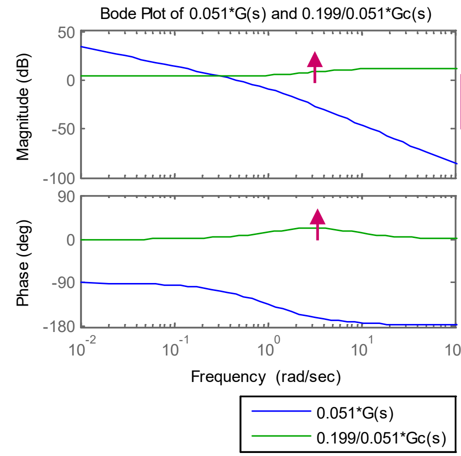


(3) Select K with zeta=0.7 for comparison

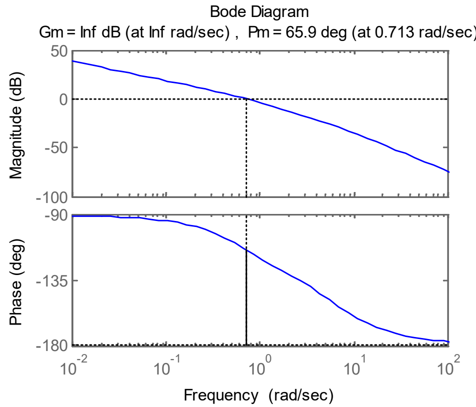
1. Step and ramp response improved
2. System speed increased

Example of system compensation with lead compensator – bode plot





Boost gain and phase



Consideration

1. To improve the speed, we need to boost the gain for higher crossover frequency
2. However, if we only boost up the gain, phase margin reduce.
3. Lead compensator can boost up the gain and phase.
4. Therefore, crossover frequency increased without changing phase margin

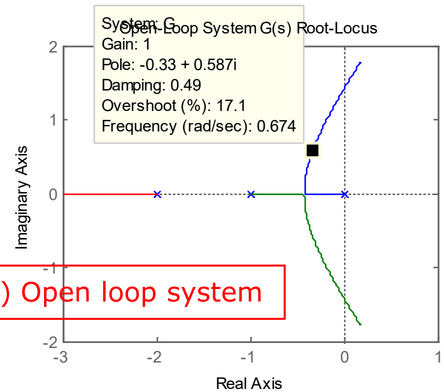
Remark:

- crossover frequency related to wn. Wn is somewhere between gain crossover freq and phase crossover freq.
- phase margin related to zeta

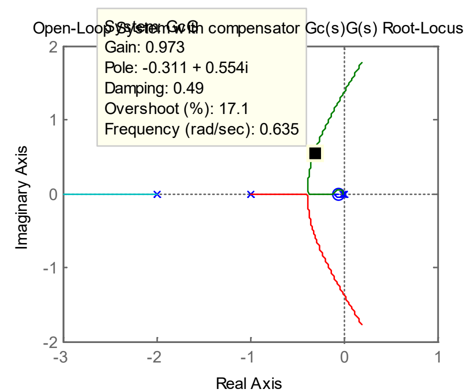
Design with lag compensator

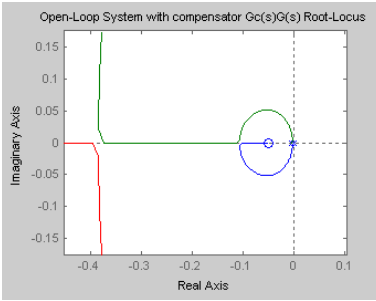
Example of system compensation with lag compensator – root locus

(2) Add a lag compensator

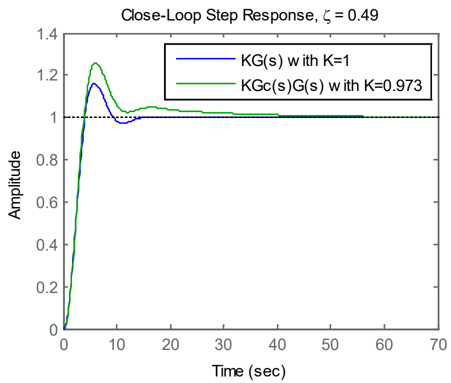


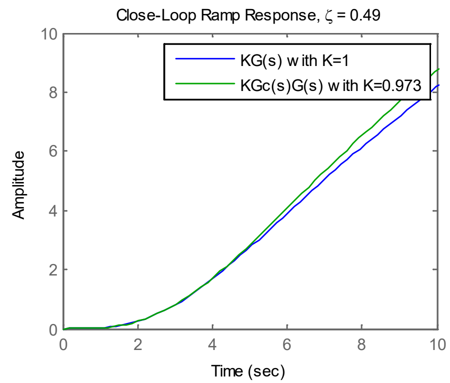
(1) Open loop system





Zoom near Origin

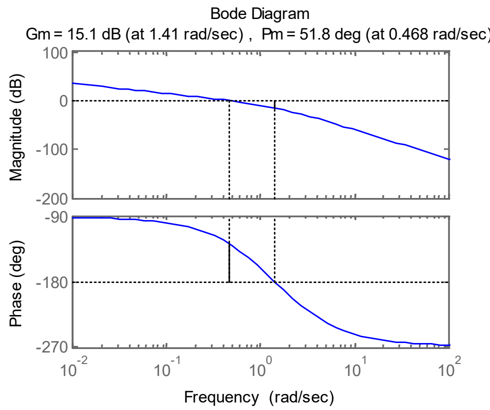


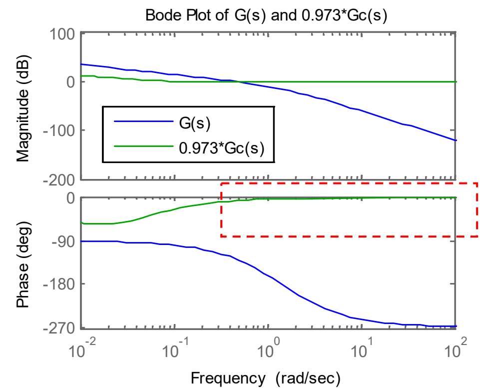


(3) Select K with zeta=0.49 for comparison

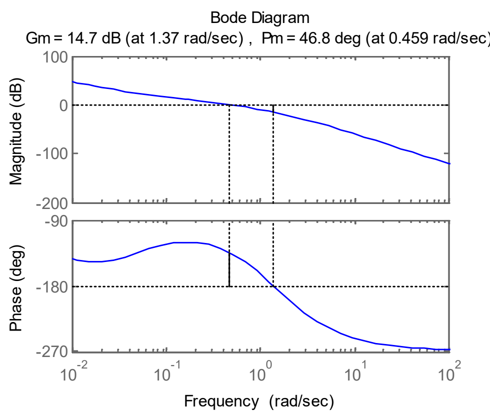
1. Avoid changing system dynamic
2. Reduce steady state error at ramp response

Example of system compensation with lag compensator – bode plot





Lag compensator doesn't affect phase in high frequency region



Consideration

1. To improve steady state response (ramp input) but avoid changing system dynamic, we need to boost the gain at low frequency without changing crossover frequency and phase margin
2. By properly select the Lag compensator, it can increase the gain at low frequency without affecting the phase near crossover frequency.
3. Therefore, low frequency gain increased without changing crossover characteristic.

Analog to Digital Implementation

Kelvin Leung, Ph.D.

24-10-2012

http://www.kskelvin.net

Analog to Digital Implementation

- Purpose
  - This ppt is intended to show the procedure of transforming an analog system into digital implementation with the help of matlab.
  - Theory of digital control is not the target of this ppt.
  - By following this ppt, you can convert your analog compensator into a digital formula and implement it in a digital processor.

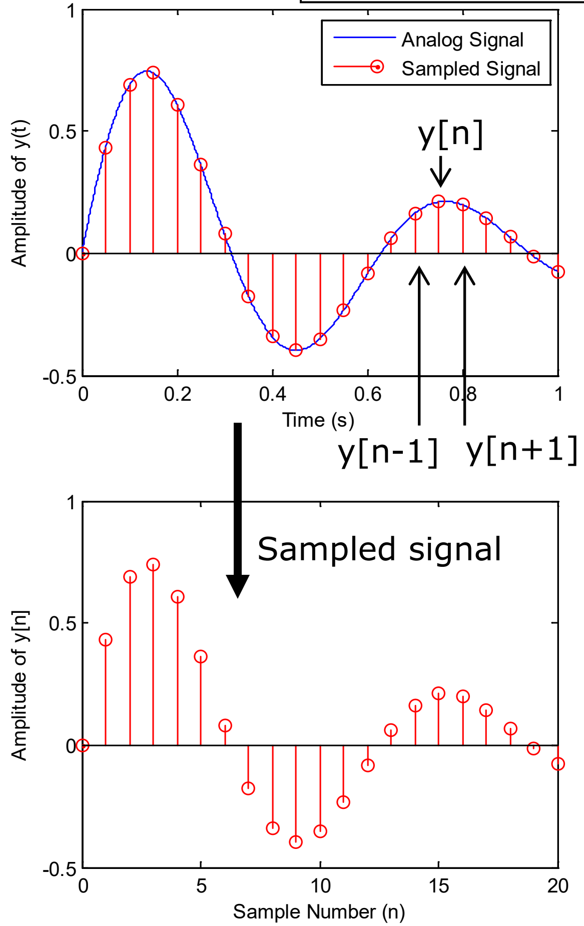
Concept of Sampling

Sampling frequency

$$f_{sampling} = \frac{1}{T_{sampling}} = \frac{1}{0.05} = 20Hz$$

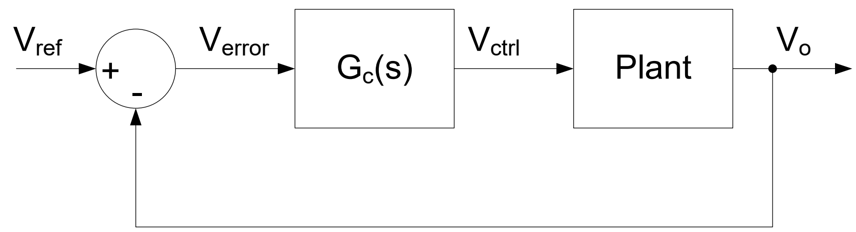
- Concept of Sampling
  - In digital controller, ADC (analog-to-digital converter) and DAC (digital-to-analog converter) are used. ADC and DAC are not continuous device but discrete time sampling input or output.
  - Sampling frequency is determined by designer. This is an important parameters to interface the analog and digital system.
- Concept of Z-transform
  - S-domain transfer function can be converts into z-domain through z-transform.
  - In z-transform, we need to remember time-shifting property

$$x[n-k] = z^{-k}X(z)$$



2nd-order transfer function example (analog transfer function)

- Assume we design a compensator $G_c(s)$ and need to implement with a digital controller



$$G_C(s) = \frac{V_{ctrl}(s)}{V_{error}(s)} = \frac{1}{s^2 + 1.4s + 1}$$

- In matlab,
  - % define a 2-nd order analog system
  - wn=1;
  - zeta=0.7;
  - num=[wn.^2];
  - den=[1 2*wn*zeta wn^2];
  - G=tf(num,den)

```
Transfer function:
        1
---------------
s^2 + 1.4 s + 1
```

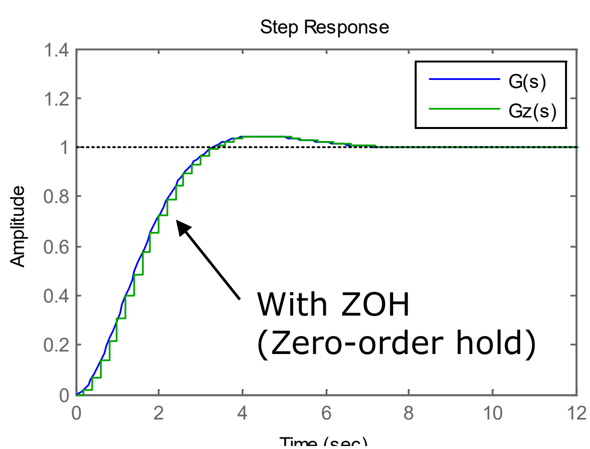
Define the analog transfer function

2nd-order transfer function example (convert to digital transfer function)

- In matlab
  - % convert the analog system to digital system
  - Gz=c2d(G,0.2) ← Sampling time
  - % form of Gz digital implementation
  - M = idpoly(Gz)
  - % plot the step response of analog and digital system
  - step(G); hold on; step(Gz); hold on;
  - legend('G(s)','Gz(s)');

```
Transfer function:
  0.0182 z + 0.01657
----------------------
z^2 - 1.721 z + 0.7558

Sampling time: 0.2
```

```
Discrete-time IDPOLY model: y(t) = [B(q)/F(q)]u(t) + e(t)
B(q) = 0.0182 q^-1 + 0.01657 q^-2

F(q) = 1 - 1.721 q^-1 + 0.7558 q^-2

This model was not estimated from data.
Sampling interval: 0.2
```



2nd-order transfer function example (digital formula from transfer function)

Method #1

Discrete - time transfer function

$$G_C(z) = \frac{v_{ctrl}(z)}{v_{error}(z)} = \frac{0.0182z + 0.01657}{z^2 - 1.721z + 0.7558}$$

$$G_C(z) = \frac{0.0182z + 0.01657}{z^2 - 1.721z + 0.7558}\frac{z^{-2}}{z^{-2}} = \frac{0.0182z^{-1} + 0.01657z^{-2}}{1 - 1.721z^{-1} + 0.7558z^{-2}}$$

Therefore,

$$v_{ctrl}(z) - 1.721z^{-1}v_{ctrl}(z) + 0.7558z^{-2}v_{ctrl}(z) = 0.0182z^{-1}v_{error}(z) + 0.01657z^{-2}v_{error}(z)$$

$$v_{ctrl}(z) = 0.0182\left(z^{-1}v_{error}(z)\right) + 0.01657\left(z^{-2}v_{error}(z)\right) + 1.721\left(z^{-1}v_{ctrl}(z)\right) - 0.7558\left(z^{-2}v_{ctrl}(z)\right)$$

$$v_{ctrl}[n] = 0.0182v_{error}[n-1] + 0.01657v_{error}[n-2] + 1.721v_{ctrl}[n-1] - 0.7558v_{ctrl}[n-2]$$

Apply time shifting property

$$x[n-k] = z^{-k}X(z)$$

```
Transfer function:
  0.0182 z + 0.01657
----------------------
z^2 - 1.721 z + 0.7558

Sampling time: 0.2
```

2nd-order transfer function example (digital formula from IDPOLY)

Method #2

Rewrite discrete - time IDPOLY as

$$y(t) = \frac{0.0182q^{-1} + 0.01657q^{-2}}{1 - 1.721q^{-1} + 0.7558q^{-2}} u(t)$$

where $y(t) \to y(k), u(t) \to u(k), q \to z$

$$y(k) = \frac{0.0182z^{-1} + 0.01657z^{-2}}{1 - 1.721z^{-1} + 0.7558z^{-2}} u(k)$$

$$y(k) = 0.0182z^{-1}u(k) + 0.01657z^{-2}u(k) + 1.721z^{-1}y(k) - 0.7558z^{-2}y(k)$$

$$y(k) = 0.0182\left(z^{-1}u(k)\right) + 0.01657\left(z^{-2}u(k)\right) + 1.721\left(z^{-1}y(k)\right) - 0.7558\left(z^{-2}y(k)\right)$$

$$y[n] = 0.0182u[n-1] + 0.01657u[n-2] + 1.721y[n-1] - 0.7558y[n-2]$$

Apply time shifting property

$$x[n-k] = z^{-k}X(z)$$

As output $y = v_{ctrl}$ and input $u = v_{error}$

$$v_{ctrl}[n] = 0.0182v_{error}[n-1] + 0.01657v_{error}[n-2] + 1.721v_{ctrl}[n-1] - 0.7558v_{ctrl}[n-2]$$

```
Discrete-time IDPOLY model: y(t) = [B(q)/F(q)]u(t) + e(t)
B(q) = 0.0182 q^-1 + 0.01657 q^-2


F(q) = 1 - 1.721 q^-1 + 0.7558 q^-2


This model was not estimated from data.
Sampling interval: 0.2
```

Formula Impementation in Matlab

- Matlab Implementation

```
% time vector
t=[0:0.2:12];    % sampling Tsampling is 0.2s
% initialization
error0=1;   % error[n]
error1=0;   % error[n-1]
error2=0;   % error[n-2]
ctrl1=0;   % ctrl[n-1]
ctrl2=0;   % ctrl[n-2]
for i=1:length(t)
   % digital implementation of Gc(z)
   ctrl0(i)=0.0182*error1+0.01657*error2+1.721*ctrl1-
     0.7558*ctrl2;

   % store time delay data for next calculation
   error2=error1;
   error1=error0;
   error=1;   % 1=step response; 0=impulse response
   ctrl2=ctrl1;
   ctrl1=ctrl0(i);
end

figure;
% plot the step response of analog and digital system
step(G); hold on; step(Gz); hold on;
% plot the response of digital implementation
plot(t,ctrl0,'ro'); hold on;
legend('G(s)','Gz(s)','Digital Implementation');
```

Initialization = 0

Calcuate Vctrl[n]